The College Of WILLIAM & MARY

Social Welfare Functions that Satisfy Pareto, Anonymity, and Neutrality, but not IIA

Donald E. Campbell
College of William and Mary

Jerry S. Kelly
Syracuse University

College of William and Mary
Department of Economics
Working Paper Number 38

August 2006

Forthcoming in *Social Choice and Welfare.*

COLLEGE OF WILLIAM AND MARY
DEPARTMENT OF ECONOMICS
WORKING PAPER # 38
August 2006

Social Welfare Functions that Satisfy Pareto, Anonymity, and Neutrality, but not IIA

Abstract

In the presence of Pareto, non-dictatorship, full domain, and transitivity, an extremely weak independence condition disallows both anonymity and neutrality.

JEL Codes: D70, D71

Keywords: Pareto, anonymity, neutrality

Donald E. Campbell
Department of Economics
College of William and Mary
Williamsburg, VA 23187-8795
decamp@wm.edu

Jerry S. Kelly
Department of Economics
Syracuse University
Syracuse, NY 13245-1090
jskelly@maxwell.syr.edu

Social Welfare Functions that Satisfy Pareto, Anonymity, and Neutrality, but not IIA

by Donald E. Campbell and Jerry S. Kelly

If the interprofile condition Independence of Irrelevant Alternatives is deleted from the assumptions of Arrow's Theorem, there do exist rules that satisfy all the remaining conditions. The best known is the (global) Borda rule which orders alternatives by the sums of the positions that alternatives have in individual preference orderings. Importantly, Borda's rule is also both neutral (symmetric with respect to alternatives) and anonymous (symmetric with respect to individuals).

But procedures like Borda's have found little place in economic theory. For example, we don't see Borda-like rules in Edgeworth-Bowley box analyses. There are at least two problems with the Borda rule. First, as defined, it works only for finite sets (to ensure that there is a topmost alternative, one in the 2^{nd} place, etc.). Second, the ordering of a pair x and y requires knowledge of all of the set of all feasible alternatives, X, and everyone's ordering of all of X; it is at an opposite extreme from satisfying Arrow's Independence of Irrelevant Alternatives. Imagine trying to even determine the set of all logically possible candidates for Congress in just one district. Imagine trying to get voters to meaningfully assign Borda counts to each such candidate. Imagine the transmission and processing problems. That we actually use very different procedures to first narrow the list of candidates to a small set X* and then apply a voting procedure to X* amounts to saying that we are unconcerned about the effects of orderings of some irrelevant alternatives - alternatives in X\X* are irrelevant.

In this paper, and in a companion paper [Campbell and Kelly (2004a)], we explore the compatibility of neutrality and/or anonymity with Arrow's conditions with a vastly weaker requirement: not all of X need be known to order some pairs of alternatives. This paper considers finite X. Infinite X is treated in the companion paper. We shall find that neutrality and anonymity interact in unexpected ways with the Pareto condition and independence conditions.

Section 0. Framework.[1]

X is the set of all ***alternatives*** or ***outcomes***. In this paper, we assume $1 < |X| < \infty$. The binary relation $\succeq$ on X is read "x is weakly preferred to y," or "x is preferred or indifferent to y." If $x \succeq y$ and $y \succeq x$ both hold we write $x \sim y$, and say that "x is indifferent to y."

A binary relation $\succeq$ on X is ***complete*** if for all x, y $\in$ X, at least one of $x \succeq y$ or $y \succeq x$ holds. Note that a complete relation is ***reflexive***, which means that $x \succeq x$ holds for each x $\in$ X. The ***asymmetric*** part of $\succeq$ is denoted by $\succ$, so $x \succ y$ if and only if $x \succeq y$ holds but $y \succeq x$ does not. When $x \succ y$ we often say that x is strongly preferred to y, or that

[1] Much of this section draws on Campbell and Kelly [2002].

x ranks strictly above y in $\succeq$. Relation $\succeq$ is ***transitive*** if for all x, y, and z in X, if $x \succeq y$ and $y \succeq z$ then $x \succeq z$; a complete and transitive relation $\succeq$ is an ***ordering***.

In this paper, we will simplify our analysis by assuming that an individual is never indifferent between distinct alternatives, in which case we say that the preference ordering is ***strong***. Formally, we say that the complete binary relation $\succeq$ is ***antisymmetric*** if for all $x, y \in X$, $x \succeq y$ and $y \succeq x$ imply $x = y$. A binary relation is a ***strong ordering*** if it is complete, transitive, and antisymmetric. Let L(X) denote the set of strong orderings on X.

The set N of individuals whose preferences are to be consulted is the (finite) set

$\{1, 2, \ldots, n\}$ with $n > 1$. A ***domain*** is some nonempty subset $\wp$ of $L(X)^N$. A member p of $L(X)^N$ is called a ***profile***, and it assigns the ordering $\succ_i^p$ to individual $i \in N$, where $\succ_i^p$ is interpreted as i's preference ordering at profile p. When p is understood, we sometimes write $\succ_i$ or R_i for $\succ_i^p$. A ***social welfare function*** for outcome set X and domain $\wp$ is a function f from $\wp$ into the set of complete binary relations on X. Social welfare functions are often called "rules." We say that rule f has ***full domain*** if $\wp = L(X)^N$. The image f(p) of f at profile p is often expressed as $\succ_{f(p)}$.

We next introduce some restrictions on the social welfare function. Given a social welfare function f on domain $\wp$, if $\succ_{f(p)}$ is transitive for each $p \in \wp$ we say that f is ***transitive-valued*** or satisfies ***transitivity***. f satisfies ***nondictatorship*** if there is no individual i such that for every p in $\wp$ and every x and y in X, $x \succ_i^p y$ implies $x \succ_{f(p)} y$. Rule f satisfies the ***Pareto criterion*** if for every $p \in \wp$ and all $x, y \in X$, we have $x \succ_{f(p)} y$ if $x \succ_i^p y$ for all $i \in N$. Rule f satisfies ***weak unanimity*** if for every $p \in \wp$ and all $x \in X$, we have $x \succ_{f(p)} y$ for all y if $x \succ_i^p y$ for all $y \neq x$ and all $i \in N$, i.e., if x is at the top of everyone's ordering, then it is at the top of $\succ_{f(p)}$.

The ***independence of irrelevant alternatives*** (IIA) condition is quite different in spirit from the Pareto criterion or nondictatorship, each of which requires a kind of responsiveness to individual preferences on the part of the social welfare function. IIA requires the social ordering of x and y to be the same at two profiles if the restrictions of those profiles to {x, y} are the same: Rule f satisfies ***IIA*** if for all $p, q \in \wp$ and all $x, y \in X$, then $p|\{x,y\} = q|\{x,y\}$ implies $f(p)|\{x,y\} = f(q)|\{x,y\}$ where $p|\{x,y\}$ and $f(p)|\{x,y\}$ are the restrictions to {x,y} of profile p and social ranking f(p) respectively.

Arrow has shown (1963) that for $|X| \geq 3$ there does not exist any transitive-valued social welfare function satisfying full domain, the Pareto condition, nondictatorship, and IIA. But if we simply delete the requirement of IIA, there are many rules satisfying the rest of Arrow's conditions plus the interprofile conditions (see Fishburn [1987]) of *neutrality* and *anonymity* which we define next.

Suppose that σ is a permutation of N. Such a permutation induces a map σ on profiles where

$$\sigma(R_1, R_2, \ldots, R_n) = (R_{\sigma(1)}, R_{\sigma(2)}, \ldots, R_{\sigma(n)}).$$

A rule f is ***anonymous*** if for every permutation σ on N and for every profile p in the domain of f, σ(p) is also in the domain and $f(\sigma(p)) = f(p)$.

Turn now from individuals to alternatives. Any permutation μ of X, the set of alternatives, induces a permutation on preference orders where $\mu(R)$ is defined by

$$\mu(x)\mu(R)\mu(y) \text{ if and only if } xRy.$$

In turn, this induces a permutation on profiles where

$$\mu(R_1, R_2, \ldots, R_n) = (\mu(R_1), \mu(R_2), \ldots, \mu(R_n)).$$

A rule f is ***neutral*** if for every profile p in the domain of f and every permutation μ on X, $\mu(p)$ is also in the domain and $f(\mu(p)) = \mu(f(p))$.

We clarify these symmetry conditions by contrasting them with other possible versions. Several authors, e.g. Sen (1970, p. 72) and Fishburn (1973, p. 161), define neutrality in such a way as to incorporate considerable independence. An informal version of this is given by Rae and Schickler (1997, p. 167):

> *Neutrality:* Suppose that all individual ordinal preferences over (x,y) are the same as they are over (w,z), then the collective outcomes over the two pairs of options must be the same.

Because we want to work in weak independence contexts, we do not use their definition.

It is also helpful to contrast our definitions with conditional versions [Campbell and Fishburn (1980)]:

> *Conditional anonymity:* For every permutation σ on N and for every profile p in thedomain $\wp$ of f, <u>if</u> $\sigma(p)$ is also in $\wp$ then $f(\sigma(p)) = f(p)$;

> *Conditional neutrality:* For every profile p in the domain $\wp$ of f and every permutation μ on X, <u>if</u> $\mu(p)$ is also in $\wp$ then $f(\mu(p)) = \mu(f(p))$.

Our (unconditional) anonymity requires that $\wp$ be closed under permutations of individuals; (unconditional) neutrality requires that $\wp$ be closed under permutations of alternatives.

Example 1: Global Borda rule [Borda (1781); for a translation and commentary, see de Grazia (1953)]. Assume $|X| = m$. On $L(X)^N$, for each individual, allocate $m-1$ points to the alternative that is ranked at the top of the individual's preference order, $m-2$ points for the alternative in second place, and in general, $m-j$ points for the alternative in j*th* position in the preference ranking. Then the social ordering is constructed by ranking x over y if and only if x's total score (summing points for x over all n individuals) is greater than y's total score. All of Arrow's conditions other than IIA are satisfied: The rule yields a transitive social ordering because the ordering "greater than" on the real numbers is transitive. It satisfies the Pareto criterion because if every individual ranks x above y then the vote total for x will exceed the vote total for y by at least n, the number of individuals. The Borda rule on $L(X)^N$ is obviously not dictatorial. Beyond Arrow's conditions, neutrality and anonymity are easily demonstrated for this rule. For ways of extending this rule to profiles of weak orderings while preserving all these properties see Luce and Raiffa (1957, p. 358), Black (1958,

p. 62), and a full development in Fishburn (1973, pp. 163-169).

In this finite case, there are many rules that share with Borda's the properties of universal domain, nondictatorship, neutrality, anonymity, and Pareto. Consider the class of weighted-score rules: For each individual i and each strong order R on finite X, there is a function u_i^R that is a numerical representation of R; that is u_i^R maps X to the reals in such a way that $u_i^R(x) > u_i^R(y)$ if and only if $x \succ y$ in R. Set $x \succeq y$ in the social ranking just when

$$\sum_{i=1}^{n} u_i^R(x) \geq \sum_{i=1}^{n} u_i^R(y)$$

Goodman and Markowitz (1952) and Fishburn (1969) provide sufficient conditions for a rule to be a weighted-score rule. A sufficient condition for anonymity is $u_i^R = u_j^R$ for all i and j. A sufficient condition for neutrality is that the numbers in the image of the representation u_i^R be the same as the image of $u_i^{R^*}$ for any two preferences R and R*.

Section 1. Neutrality

We are going to delete IIA from Arrow's list but still impose very weak independence conditions and determine if there are rules satisfying neutrality or anonymity. The first independence condition is from Campbell and Kelly (2000):

Independence of Some Alternatives (ISA): For every pair of alternatives x and y in X there is a proper subset Y of X such that for any two profiles p and p′ in the domain, if $p|Y = p'|Y$ then $f(p)|\{x,y\} = f(p')|\{x,y\}$.

An even less demanding condition is

Weakest Independence (WI): For at least one pair of alternatives x and y in X there is a proper subset Y of X such that for any two profiles p and p′ in the domain, if $p|Y = p'|Y$ then $f(p)|\{x,y\} = f(p')|\{x,y\}$.

The Borda Rule violates WI. Accordingly, any list of properties that is sufficient for that rule, if supplemented by WI, would yield an impossibility result. While Young (1974), Hansson and Sahlquist (1976), Coughlin (1979/80), Nitzan and Rubinstein (1981), and Debord (1992) have characterizations of Borda's rule, they work in a different context than ours; for these authors, a rule has to work for a variable number of individuals. We do not know of any similar interesting characterizations of Borda's rule with a fixed set of individuals.

All existing examples of rules satisfying ISA and Arrow's conditions other than IIA, like the gateau rules [Campbell and Kelly (2000)], are far from neutral or anonymous. To explore the possibility of finding rules that satisfy independence conditions weaker than IIA and are either neutral or anonymous, we introduce some terminology and one important principle. Given a rule f and a subset Y of X, we say that Y is ***sufficient*** for {x,y} if for any two profiles p and p′ in the domain, $f(p)|\{x,y\} = f(p')|\{x,y\}$ if $p|Y = p'|Y$. If Y is sufficient for {x,y} and $Y \subseteq Z \subseteq X$, then clearly Z is also sufficient for {x,y}. The family of sufficient sets can place substantial restrictions on the possible departures from IIA, as the following ***intersection principle*** shows. It is important to note that it does not assume finiteness of X, or the Pareto criterion, or any independence condition, or any type of transitivity property for f(p).

Intersection principle: [Campbell and Kelly (2000)] If the domain of f is $L(X)^N$, and Y and Z are each sufficient for {x,y} then $Y \cap Z$ is sufficient for {x,y}.

For the case of finite X and for each pair {x,y} of distinct alternatives, the intersection principle ensures the existence of a smallest set sufficient for {x,y} — a sufficient set that is a subset of every set sufficient for {x,y}. Such a smallest set sufficient for {x,y} is the ***relevant set*** for {x,y} and is denoted by $\psi(\{x,y\})$ or $\psi(x,y)$. ψ is the ***relevance map*** associated with f.

If $\{x,y\} \subseteq \psi(x,y)$ for all distinct x and y, we say the social welfare function is ***natural*** (we also say then that ψ is natural). It is shown in Campbell and Kelly (2000) that any social welfare function satisfying full domain and Pareto is natural. We will need a strengthening of that:

Lemma 2. If the domain of f includes $L(X)^N$ and f satisfies weak unanimity, then f is natural.

Proof: Let r be a profile such that for every i, r(i) has x on top and y in second place. Profile r′ is obtained from r by interchanging x and y in each ordering. f(r) has $x \succ^r y$ while f(r′) has $y \succ^{r'} x$. But if S is any subset of X that does not contain both x and y, $r|S = r'|S$. □

Of course this proof requires much less than a full domain assumption. It would be sufficient to require only the "free pairs at top" condition of Aswal, Chatterji, and Sen (1999).

We now show that among the rules satisfying weakest independence, there are no Borda-like rules; that is, there are no social welfare functions satisfying all of transitivity, full domain, the Pareto criterion, nondictatorship, neutrality and anonymity. In fact, we can weaken anonymity to non-dictatorship for this result and we can weaken Pareto to weak unanimity. We show that full domain, weak unanimity, weakest independence, and neutrality imply IIA. Then weak unanimity implies Pareto. In the presence of transitivity, Arrow's Theorem then implies the existence of a dictator.

Theorem 3. If X is finite with $|X| \geq 3$, there does *not* exist a social welfare function satisfying full domain, transitivity, weak unanimity, nondictatorship, weakest independence, and neutrality.

Theorem 3 will follow quickly from two lemmas.

Lemma 4. If X is finite with $|X| \geq 3$ and f satisfies weakest independence and neutrality, then f also satisfies ISA.

Proof: Since f satisfies weakest independence, there exist x, y, and z in X such that $X\backslash\{z\}$ is sufficient for {x,y}. Given any pair u, v in X, let θ be a permutation that maps x to u and y to v. Let $w = \theta(z)$. We show $X\backslash\{w\}$ is sufficient for {u,v}. Otherwise there would be profiles r and r* such that

$$r|_{X\backslash\{w\}} = r^*|_{X\backslash\{w\}}$$

with $u \succeq v$ at r and $v \succ u$ at r*. Then consider the profiles $\theta^{-1}(r)$ and $\theta^{-1}(r^*)$. Since $r|_{X\backslash\{w\}} = r^*|_{X\backslash\{w\}}$, $\theta^{-1}(r)|_{X\backslash\{z\}} =$

$\theta^{-1}(r^*)|_{X\setminus\{z\}}$. By neutrality, x $\succeq$ y at $\theta^{-1}(r)$ and y $\succ$ x at $\theta^{-1}(r^*)$, a violation of the sufficiency of X\{z} for {x,y}. □

Lemma 5. If X is finite with |X| ≥ 3 and f satisfies full domain, weak unanimity, weakest independence, and neutrality, then f also satisfies IIA.

Proof: By Lemma 4, f satisfies ISA. Let x and y be any two distinct alternatives in X. By ISA, there exists a z in X such that X\{z} is sufficient for {x,y}. By Lemma 2, {x,y} ⊆ ψ(x,y) so z cannot be x or y; that is, there is a z in X\{x,y} such that X\{z} is sufficient for {x,y}. Then, by neutrality, for every z in X\{x,y} the set X\{z} is sufficient for {x,y}. By finiteness of X and repeated application of the intersection principle,

$$\{x,y\} = \bigcap_{z \in X \setminus \{x,y\}} X \setminus \{z\}$$

is sufficient for {x,y}, i.e., IIA holds. □

Proof of Theorem 3: By Lemma 5, f satisfies IIA. By IIA and weak unanimity, f satisfies Pareto. But by Arrow's Theorem, f cannot then also satisfy all of full domain, transitivity, and non-dictatorship. □

We can use Lemmas 4 and 5 even without a full domain. For example, a set of axioms that is necessary and sufficient for majority rule on a domain of single-peaked preferences will still characterize majority rule when we replace Pareto by weakest unanimity and IIA by weakest independence, provided that neutrality is one of the original axioms.

It is straightforward to show that if any of the conditions in Theorem 3 is dropped, the remaining conditions are consistent. If |X| = 2, the remaining conditions are satisfied by simple majority voting. If Pareto is dropped, the remaining conditions are satisfied by the null rule that at every profile u, assigns the social ranking that sets every two alternatives indifferent. If transitive-valuedness is dropped, the remaining conditions are satisfied by simple majority voting. If weakest independence is dropped, the remaining conditions are satisfied by the global Borda rule. If nondictatorship is dropped, the remaining conditions are satisfied by the rule that sets the social ranking to be the same as the first individual's ordering. If full domain is dropped, the remaining conditions are satisfied by simple majority voting on the domain of single-peaked profiles.

If neutrality is dropped, the remaining conditions can be satisfied as seen in Example 6 below. In fact the rest can be satisfied if neutrality is only weakened to *impartiality*. For a pair of alternatives, x and y, we say a rule is ***impartial*** between x and y if for every profile q in ℘, when profile q* is obtained from q by (only) transposing x and y in everyone's ordering, then q* is in ℘ and x $\succeq$ y implies y $\succeq^*$ x. If a rule is neutral, it is impartial between any two alternatives, but — as will be shown — a rule can be impartial between any two alternatives without being neutral. Neutrality requires that if profile r′ is obtained from r by transposing x and y in everyone's preference ordering, then

(1) The social ranking of x and y is reversed;

(2) For each w, we have x $\succeq'$ w if and only if y $\succeq$ w;

(3) The rest of the social ranking remains unchanged.

That is, neutrality incorporates a great deal of independence. But in our framework with weakened independence, if x and y are interchanged in everyone's ordering, why should that not possibly affect the social ranking of other alternatives? Finally, it is important to note that the impartiality relation is not transitive.

Example 6: To examine the consequences of dropping neutrality, let X = {x,y,z,w} and let f be a standard gateau rule [Campbell and Kelly (2000)] defined as follows: Rank {x,y} as well as {x,z} and {x,w} by simple majority rule. Include any additional ranking required by transitivity, e.g., if y $\succ$ x and x $\succ$ w, set y $\succ$ w. Finally within the set of alternatives ranked above (or below) x, socially rank as in the first individual's preference ordering. This rule satisfies ISA (the relevant set for any pair {a,b} is {a,b,x}), full domain, transitive-valuedness, and, for extra measure, even Pareto. The rule is not neutral, for consider the profile

1	2	3
x	z	z
y	w	w
w	x	x
z	y	y

\where the rule yields w $\succ$ z $\succ$ x $\succ$ y. Now create a new profile by interchanging x and w for everyone

1	2	3
w	z	z
y	x	x
x	w	w
z	y	y

Neutrality would force x $\succ'$ z $\succ'$ w $\succ'$ y, by interchanging x and w in the social ranking. However, this rule actually yields z $\succ'$ x $\succ'$ w $\succ'$ y.

We now show the rule of this example is impartial on any pair of alternatives. Suppose that α $\succ$ x, so α defeats x by simple majority vote. Then if x and α are interchanged in every individual preference ordering, x will defeat α by simple majority vote and x $\succ$ α. Now suppose x $\notin$ {α, β}. If α $\succ$ β because α $\succ$ x $\succ$ β, then interchanging α and β will now make β defeat x and x defeat α by simple majority vote. Finally if, say, α $\succ$ β because both defeat x by simple majority vote and $\alpha \succ_1 \beta$, then interchanging α and β leaves both still defeating x by simple majority vote and then $\beta \succ^* \alpha$ because $\beta \succ_1{}^* \alpha$. This example shows that Theorem 3 does not go through if neutrality is relaxed to impartiality on all pairs (even if weakest independence is strengthened to ISA).

Notice that the rule in Example 6 violates anonymity. We conjecture that if X is finite and f on $L(X)^N$ is transitive-valued, anonymous, and satisfies ISA, then impartiality on all pairs of alternatives implies neutrality.[2] Certainly if |X| = 3 then ISA implies IIA and then impartiality implies neutrality. Also impartiality on all pairs of alternatives implies non-imposition. Wilson's theorem (1972), then tells us that f must be either dictatorial, inversely dictatorial, or null, i.e., at every profile assigns the social ranking that exhibits indifference between any pair of alternatives. Since anonymity rules out both dictatorship and inverse dictatorship, the rule must be null, which is obviously neutral. In general, any counterexample to this conjecture must violate IIA.

Section 2. Anonymity

In the previous section, we saw that in the finite X case, there is no neutral rule satisfying the rest of Arrow's conditions other than IIA if we impose even a very weak independence condition. We next explore how this conclusion is affected when anonymity replaces neutrality. The analysis turns out to be considerably more subtle. Of course, since we are going to require anonymity, we do not also have to require nondictatorship.

Conjecture 7. For any X, there does not exist a rule satisfying all of: ISA, Pareto, full domain, transitivity, and anonymity.

We can contrast Conjecture 7 with Arrow's Theorem from the standpoint of domains on which it may hold. If Arrow's Theorem holds on some domain it may still fail on an expanded domain. An example is provided by Bordes and Le Breton (1990); a classification of such possibilities is given in Kelly (1994). The problem is that non-dictatorship may hold on a domain but not on a subdomain. But the conditions of Conjecture 7 (except, of course, full domain) all have the property that if they hold for a rule on a large domain, those properties are inherited on any subdomain. Therefore, if the conjecture is true on some particular domain it is also true on any superset domain.

While we do not have a general proof of Conjecture 7, we do have partial results. First we have a complete proof for the case m = 4 and an even number of individuals [Campbell and Kelly (2004b)]. Second, we can establish the result for a very broad class of relevance maps. Social welfare rules that are not unusually contrived have relevance maps that are well-behaved in the following sense. A social welfare function f with relevance map ψ is ***simple*** if $\{x,y\} \subseteq \psi(u,v)$ implies $\psi(x,y) \subseteq \psi(u,v)$ [Campbell and Kelly (2000)]. We will prove that if there are at least two alternatives then there does not exist a simple social welfare function satisfying full domain, transitivity, Pareto, ISA, and anonymity.

---

[2] A referee points out that if Conjecture 7 in the next section is true, the claim here would have limited scope as f would have to violate Pareto.

Actually our proof will work for an even broader class of rules than the simple ones, so we introduce that class next. A triple {x,y,z} satisfies ***exclusion*** if $x \notin \psi(y,z)$, $y \notin \psi(x,z)$, and $z \notin \psi(x,y)$. We also say ψ satisfies exclusion if there is at least one triple satisfying exclusion; and a social welfare rule satisfies exclusion if its associated relevance map satisfies exclusion.

Theorem 8. For finite X, any natural rule f that satisfies ISA and simplicity also satisfies exclusion.

Proof: Since X is finite, there exists a pair x, y, with maximal relevant set, i.e., maximal with respect to inclusion: there is no pair w, z, such that ψ(x,y) is a proper subset of ψ(w,z). By ISA, there is an alternative $z \notin \psi(x,y)$. Consider ψ(x,z); since f is natural, $z \in \psi(x,z)$, so $\psi(x,z) \neq \psi(x,y)$. Thus if $\psi(x,y) \subset \psi(x,z)$, it would be a proper subset, contrary to our choice of x and y. But if $y \in \psi(x,z)$, simplicity <u>would</u> give $\psi(x,y) \subset \psi(x,z)$. Therefore $y \notin \psi(x,z)$. Similarly, $x \notin \psi(y,z)$. Thus {x,y,z} satisfies exclusion. □

It's difficult to find examples of transitive-valued social welfare functions that aren't simple. Any rule that satisfies either of the extreme independence assumptions — IIA on the one hand and ψ(x,y) = X for all distinct x and y on the other — is simple.

The conclusion of Theorem 8 need not hold for simple rules that don't satisfy ISA. Suppose $X = \{x_1, x_2, ..., x_m\}$; there exist rules with relevance map

$$\psi(x_i, x_j) = \{x_1, x_2, ..., x_k\}$$

where $k = \max \{x_i, x_j\}$. Such a rule is natural and simple, but violates exclusion (and ISA).

Also the conclusion of the theorem need not hold for simple rules that aren't natural. Again suppose X = $\{x_1, x_2, ..., x_m\}$ with $z \in X$, but consider the rule that sets $x_i \succ x_j$ if anyone has any alternative from X\{z} between x_i and x_j and sets $x_j \succ x_i$ otherwise. The relevance map is $\psi(x_i, x_j) = X\setminus\{z\}$ and this satisfies simplicity but not exclusion.

Exclusion is strictly weaker than simplicity in the presence of full domain, Pareto, and ISA, i.e., there exist rules satisfying exclusion but not simplicity. To see this, first we exhibit a relevance map that satisfies exclusion but not simplicity. With X = {x,y,z,w}, consider ψ:

S	ψ(S)
{x,y}	{x,y,z}
{y,z}	{y,z,w}
{x,w}	{x,w,z}
{y,w}	{y,w,z}
{z,w}	{z,w}
{x,z}	{x,z,w}

{x,y,w} is an exclusion triple; but while {y,z} $\subset$ ψ(x,y), it is not true that ψ(y,z) $\subset$ ψ(x,y).

Then apply

Lemma 9. For any given natural map ψ from pairs of alternatives to subsets of X, there exists a Pareto, anonymous rule f on $L(X)^N$ having ψ as its associated relevance map.

Proof: Define f at profile r as follows: Take any given strong order L of X; the ranking of x and y by f(r) is given by applying one of the conditions:

(1) If no element of ψ(x,y)\{x,y} is between x and y in anyone's ordering at r, then rank x and y by simple majority vote;

(2) If for even one individual even one alternative in ψ(x,y)\{x,y} is between x and y, then let α be the element from {x,y} that is higher in L and let β be the lower; f(r) ranks α above β unless everyone prefers β to α, in which case f(r) ranks β above α. □

We now present our main result on the non-existence of good anonymous rules satisfying ISA. For countable X, the analog of this result does not hold (Campbell and Kelly, 2004a).

Theorem 10. For any finite X, there does not exist a rule satisfying all of exclusion, ISA, Pareto, full domain, transitivity, and anonymity.

Proof: We start by describing a way of constructing new social welfare functions from old. Given a rule f on $L(X)^N$, a subset S of X, and a strong ordering ℓ on S, let X* = X\S and construct f* on $L(X^*)^n$ as follows. For each r in $L(X^*)^n$ construct profile r′ in $L(X)^N$ by extending each r(i) by putting each member of S below all of X\S, with S ordered as ℓ. Then

$$f^*(r) = f(r')|X^*.$$

We then say f* is ***induced from f by X**** (and ℓ). We have defined f* to satisfy full domain. Also it is immediate that for each of the following properties, f* has the property if f does:

(i) transitive-valued;

(ii) weak unanimity; and

(iii) anonymity.

Next we need the observation that for every x,y in X*

$$\psi^*(x,y) \subseteq \psi(x,y) \cap X^* \qquad (1)$$

where ψ^* is the relevance map for f*. For if profiles r and q in $L(X^*)^n$ agree on $\psi(x,y) \cap X^*$, then r′ and q′ agree on $\psi(x,y)$, so f would order x and y the same at r′ and q′; hence f* would order x and y the same at r and q. Note that $\psi^*(x,y)$ may be a proper subset of $\psi(x,y) \cap X^*$.

Finally, let {x,y,z} be an exclusion triple. Let ℓ^* be a strong order on $S^* = X\setminus\{x,y,z\}$

and construct the f* induced from f and ℓ^*. f* satisfies full domain, transitivity, and Pareto. Now {x,y,z} is an exclusion triple, so $z \notin \psi(x,y)$. Since $\psi^*(x,y) \subseteq \psi(x,y) \cap \{x,y,z\}$— by (1) above — we have $\psi^*(x,y) \subseteq \{x,y\}$. A similar argument works for each pair in {x,y,z}, so f* satisfies IIA. By Arrow's Theorem, f* is dictatorial and so fails anonymity. Therefore f also fails anonymity. □

We have shown much more than a failure of anonymity. We have shown that for any rule satisfying all of exclusion, ISA, Pareto, full domain, and transitivity, there are large subdomains on which one (and only one) individual is a dictator. Instead of using the Pareto criterion and Arrow's Theorem, we could invoke Wilson's Theorem and get a violation of anonymity by introducing a condition to ensure that f is not null and satisfies non-imposition on the sub-family of profiles. We don't do that formally because the proof is transparent enough, and the conditions would be very contrived. The reader can see that it wouldn't take much to extend the theorem.

While the proof indicates that the properties of transitive-valuedness, weak unanimity, and anonymity are inherited by f* from f, questions about inheritance of independence are more difficult. If f satisfies ISA, that need not be true of f*.

Example 11. With n odd and X = {a,b,x,y,z}, f(r) is partially described as

$$T \succ \alpha \succ M \succ \beta \succ B$$

where

α is the majority winner between a and b;

β is the majority loser;

T is the set of alternatives γ such that everyone prefers γ to α;

B is the set of alternatives γ such that everyone prefers β to γ; and

M is everything else.

f(r) is completed by ordering T as #1 does; ordering M as #2 does; and ordering B as #3 does. For any pair {u,v} in X, $\psi(u,v) = \{u,v\} \cup \{a,b\}$ so ISA holds for f, which also satisfies full domain, transitivity, weak unanimity, and simplicity (but not, of course, anonymity). Now let f* be induced by $X^* = X\backslash\{x\}$. Then $\psi^*(y,z) = \{y,z,a,b\}$ so f* fails ISA.

If any of the conditions in Theorem 10 – other than exclusion – is dropped, the remaining conditions are consistent. If anonymity is dropped, the remaining conditions are satisfied by complete dictatorship, e.g., at any profile the social ranking is identical with the preference ordering of the first individual. If Pareto is dropped, the remaining conditions are satisfied by the rule that at every profile u, assigns the social ranking that sets every two alternatives indifferent. If transitive-valuedness is dropped, the remaining conditions are satisfied by simple majority voting. If full-domain is dropped, the remaining conditions are satisfied by majority voting on the profiles satisfying single-peakedness. In all of the examples so far in this paragraph, not only is ISA satisfied, so is IIA. If ISA is dropped, the remaining conditions are satisfied by the global Borda rule.

Of course there are relevance maps that fail exclusion but still satisfy ISA:

S	ψ(S)
{x,y}	{x,y,z}
{y,z}	{y,z,w}
{x,w}	{x,w,y}
{y,w}	{y,w,x}
{z,w}	{z,w,x}
{x,z}	{x,y,z}

Notice there are pairs (in this case, <u>all</u> pairs) with relevant sets that contain all but one alternative. All of the difficulties one can imagine with determining all of X and with gathering individual orderings on all of X surely also apply to each determining all of X\{x} and gathering orderings on that set. For this reason, the following ***Size Conjecture*** is important: Given finite X, a natural rule f on X satisfying full domain, transitivity, and Pareto, with associated relevance map ψ, that is not exclusive must have very large ψ images. There may need to exist at least one pair of alternatives a,b with $|\psi(a,b)| \geq |X| - 1$.

There is some small |X| support for the Size Conjecture. If $|X| = 4$, suppose to the contrary that for every pair of alternatives a,b the relevance map gives $|\psi(a,b)| \leq 2$. Then the natural property implies $\{a,b\} \subset \psi(a,b)$ and so $|\psi(a,b)| \leq 2$ implies $\psi(a,b) = \{a,b\}$ which shows every triple is exclusive.

Further evidence related to the Size Conjecture is the following result [Graver (2004)]. If a rule has relevance map ψ that is natural, then if it fails exclusion, the average ψ image must be large:

$$\frac{1}{\binom{m}{2}} \sum_{\{x,y\}} |\psi(x,y)| \geq \frac{m+4}{3}$$

Larger bounds are expected to follow if we also impose some of full domain, Pareto, and transitivity.

References

Arrow, K. J. (1963) *Social Choice and Individual Values* (2nd ed., New York: Wiley).

Aswal, N., S. Chatterji, and A. Sen (2003) 'Dictatorial Domains,' *Economic Theory* **22**: 45-62.

Black D. (1958) *The Theory of Committees and Elections* (London and New York: Cambridge University Press).

Borda, J.-C. (1781) 'Mémoire sur les Élections au Scrutin,' *Histoire de l'Académie Royale des Sciences*.

Bordes, G. A. and M. Le Breton (1990) 'Arrovian Theorems for Economic Domains: The Case Where There are Simultaneously Private and Public Goods,' *Social Choice and Welfare* 7:1-17.

Campbell, D. E. and Fishburn, P. C. (1980) 'Anonymity Conditions in Social Choice Theory,' *Theory and Decision* 12: 21-39.

Campbell, D. E. and J. S. Kelly (2000) 'Information and Preference Aggregation,' *Social Choice and Welfare* 17: 3-24.

Campbell, D. E. and J. S. Kelly (2002) 'Impossibility Theorems in the Arrovian Framework' *Handbook of Social Choice and Welfare*, edited by K. J. Arrow, A. K. Sen, and K. Suzumura (Amsterdam: North-Holland) 35-94.

Campbell, D. E. and J. S. Kelly (2004a) 'Pareto, Anonymity, and Neutrality, but not IIA: The Countable Case' .

Campbell, D. E. and J. S. Kelly (2004b) 'Pareto, Anonymity, Independence, and Neutrality: Four Alternatives'.

Coughlin, P. J. (1979/80) 'A Direct Characterization of Black's First Borda Count,' *Economics Letters* 4: 131-133.

Debord, B. (1992) 'An Axiomatic Characterization of Borda's k-choice Function,' *Social Choice* *and Welfare* 9: 337-343.

De Grazia, A. (1953) ‘Mathematical Derivation of an Electoral System,’ *Isis* 44: 42-51.

Fishburn, P. C. (1969) ‘Preferences, Summation and Social Welfare Functions,’ *Management Science* 16: 179-186.

Fishburn, P. C. (1973) *The Theory of Social Choice* (Princeton: Princeton University Press).

Fishburn, P. C. (1987) *Interprofile Conditions and Impossibility* (Chur, Switzerland: Harwood Academic Publishers).

Goodman, L. A. and H. Markowitz (1952) ‘Social Welfare Functions Based on Individual Rankings,’ *American Journal of Sociology* 58: 257-262.

Graver, J. (2004) ‘A Note on Natural Correspondences that Satisfy Exclusion’.

Hansson, B. and H. Sahlquist (1976) ‘A Proof Technique for Social Choice with Variable Electorate,’ *Journal of Economic Theory* 13: 193-200.

Kelly, J. S. (1994) ‘The Bordes-Le Breton Exceptional Case,’ *Social Choice and Welfare* 11: 273-281.

Luce, R. D. and H. Raiffa (1957) *Games and Decisions* (New York: Wiley).

Nitzan, S. I. and A. Rubinstein (1981) ‘A Further Characterization of Borda Ranking Method,’ *Public Choice* 36: 153-158.

Rae, D. W. and E. Schickler (1997) ‘Majority Rule,’ in *Perspectives on Public Choice* edited by D. C. Mueller (Cambridge: Cambridge University Press) 163-180.

Sen, A. K. (1970) *Collective Choice and Social Welfare* (San Francisco: Holden Day).

Young, H. P. (1974) ‘An Axiomatization of Borda's Rule,’ *Journal of Economic Theory* 9: 43-52.

Wilson, R. B. (1972) ‘Social Choice Without the Pareto Principle,’ *Journal of Economic Theory* 5: 478-486.