Social Welfare Functions that Satisfy Pareto, Anonymity, and Neutrality: Countable Many Alternatives

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Abstract

For a finite number of alternatives, in the presence of Pareto, non-dictatorship, full domain, and transitivity, an extremely weak independence condition is incompatible with each of anonymity and neutrality (Campbell and Kelly [2006]). This paper explores how those results are affected when there are countably many alternatives.

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Donald E. Campbell and Jerry S. Kelly

We would like to express our indebtedness to Peter Fishburn for his pioneering work in social choice theory and our pleasure in co-authoring with him[1]. More specific to this paper, Peter had early doubts about rules that required preference information on all alternatives in order to socially rank two alternatives. Addressing independence he writes (Fishburn [1973])

> If in fact the social choice can depend on infeasibles, which infeasibles should be used? For with one set of infeasibles, feasible x might be the social choice, whereas feasible y ≠ x might be the social choice if some other infeasible set were adjoined to [feasible set] Y. Hence, the idea of allowing infeasible alternatives to influence the social choice introduces a potential ambiguity into the choice process that can be at least alleviated by insisting on the independence condition.

Campbell and Kelly [2000] provides a formal answer to Fishburn's question by defining and exploring "relevance sets." Fishburn continues:

> This obviously ties into the choice of the universal set X of alternatives in a particular situation. If independence is adopted, then the contents of X are not especially important as long as they include, at least conceptually, anything that might qualify as a feasible candidate or alternative. If independence is not adopted, the ambiguity noted in the preceding paragraph may cause significant problems in attempting to justify just what should and should not be included in X.

For a finite number of alternatives, Campbell and Kelly [2006] have shown that in the presence

[1] Campbell and Fishburn (1980); Fishburn and Kelly (1997).

of Pareto, non-dictatorship, full domain, and transitivity, an extremely weak interprofile condition [Fishburn (1987)] is incompatible with each of anonymity and neutrality. This paper explores how those results are affected when there are countably many alternatives.

We will show that there do exist neutral rules that satisfy all of Arrow's conditions except IIA and also anonymous rules that satisfy all of Arrow's conditions except IIA. There also exist anonymous rules that satisfy all of Arrow's conditions except IIA, and for which the social ordering on a pair depends only on the individual preferences restricted to a finite set. We show that there do not exist neutral rules for which the social ordering on even one pair depends only on the individual preferences restricted to a proper subset of the outcome space. And that result will not require either transitivity of the social ordering or a Pareto condition.

Section 1. Framework.[2]

X is the set of all ***alternatives*** or ***outcomes***. In this paper, we assume X has countably many elements. The binary relation $\succeq$ on X is read "x is weakly preferred to y," or "x is preferred or indifferent to y."

A binary relation $\succeq$ on X is ***complete*** if for all x, y $\in$ X, either $x \succeq y$ or $y \succeq x$ holds. Note that a complete relation is ***reflexive***, which means that $x \succeq x$ holds for each x $\in$ X. The ***asymmetric*** part of $\succeq$ is denoted by $\succ$, so $x \succ y$ if and only if $x \succeq y$ holds but $y \succeq x$ does not. When $x \succ y$ we often say that x is strongly preferred to y, or that x ranks strictly above y in $\succeq$. Relation $\succeq$ is ***transitive*** if for all x, y, and z in X, if $x \succeq y$ and $y \succeq z$ then $x \succeq z$; a complete and transitive relation $\succeq$ is an ***ordering***.

In this paper, we will simplify our analysis by assuming that an individual is never indifferent between distinct alternatives, in which case we say that the preference ordering is ***strong***. Formally, we say that the complete binary relation $\succeq$ is ***antisymmetric*** if for all x, y $\in$ X, $x \succeq y$ and $y \succeq x$ imply x = y. A binary relation is a ***strong ordering*** if it is complete, transitive, and antisymmetric. Let L(X) denote the set of strong orderings on X.

The set N of individuals whose preferences are to be consulted is the (finite) set

{1, 2, . . . , n} with n > 1. A ***domain*** is some nonempty subset $\wp$ of $L(X)^N$. A member p of $L(X)^N$ is called a ***profile***, and it assigns the ordering p(i) to individual i $\in$ N. We typically write $x \succ_i^p y$ to indicate that individual i strictly prefers x to y in ordering p(i). When p is understood, we sometimes write $\succ_i$ for

[2] Much of this section is drawn from Campbell and Kelly [2006].

$\succ_i^p$. A ***social welfare function*** for outcome set X and domain $\wp$ is a function f from $\wp$ into the set of complete binary relations on X. Social welfare functions are often called "rules." We say that rule f has ***full domain*** if $\wp = L(X)^N$. If x is ranked higher than y at the image f(p) of f at profile p we write $x \succ_{f(p)} y$.

We next introduce some restrictions on the social welfare function f on domain $\wp$: If f(p) is transitive for each $p \in \wp$ we say that f is ***transitive-valued*** or satisfies ***transitivity***. The rule f satisfies ***nondictatorship*** if there is no individual i such that for every p in $\wp$ and every x and y in X, $x \succ_i^p y$ implies $x \succ_{f(p)} y$. Rule f satisfies the ***Pareto criterion*** if for every $p \in \wp$ and all $x, y \in X$, we have $x \succ_{f(p)} y$ if $x \succ_i^p y$ for all $i \in N$. Rule f satisfies ***weak unanimity*** if for every $p \in \wp$ and all $x \in X$, we have $x \succ_{f(p)} y$ for all $y \neq x$ <u>if</u> $x \succ_i^p y$ for all $y \neq x$ and all $i \in N$. In words, if x is at the top of everyone's ordering at p, then it is at the top of f(p).

The ***independence of irrelevant alternatives*** (IIA) condition is quite different in spirit from the Pareto criterion or nondictatorship, each of which requires a kind of responsiveness to individual preferences on the part of the social welfare function. IIA requires the social ordering of x and y to be the same at two profiles if the restrictions of those profiles to {x, y} are the same: Formally, rule f satisfies ***IIA*** if for all $p, q \in \wp$ and all $x, y \in X$, $p|\{x,y\} = q|\{x,y\}$ implies $f(p)|\{x,y\} = f(q)|\{x,y\}$, where $p|\{x,y\}$ and $f(p)|\{x,y\}$ are the restrictions to {x,y} of profile p and social ranking f(p) respectively.

We will also need some weaker versions of independence:

Independence of Some Alternatives (ISA): For <u>every pair</u> of alternatives x and y in X there is a <u>proper</u> subset Y of X such that for any two profiles p and p′ in the domain, if $p|Y = p'|Y$ then $f(p)|\{x,y\} = f(p')|\{x,y\}$.

Weakest Independence (WI): For <u>at least one pair</u> of alternatives x and y in X there is a <u>proper</u> subset Y of X such that for any two profiles p and p′ in the domain, if $p|Y = p'|Y$ then $f(p)|\{x,y\} = f(p')|\{x,y\}$.

The modified independence conditions suggest some new terminology: Given a rule f and a subset Y of X, we say that Y is ***sufficient*** for {x,y} if for any two profiles p and p′ in the domain, $f(p)|\{x,y\} = f(p')|\{x,y\}$ if $p|Y = p'|Y$. If Y is sufficient for {x,y} and $Y \subseteq Z \subseteq X$, then clearly Z is also sufficient for {x,y}. The family of sufficient sets can place substantial restrictions on the possible departures from IIA, as the following ***intersection principle*** shows. It is important to note that it does not assume finiteness of X, or the Pareto criterion, or any independence condition, or any type of transitivity property for f(p).

Intersection principle: [Campbell and Kelly (2000)] If the domain of f is $L(X)^N$, and Y and Z are each sufficient for {x,y} then $Y \cap Z$ is sufficient for {x,y}.

For the case of finite X and for each pair {x,y} of distinct alternatives, the intersection principle ensures the existence of a smallest set sufficient for {x,y} — a sufficient set that is a subset of every set sufficient for {x,y}. Such a smallest set sufficient for {x,y} is the ***relevant set*** for {x,y} and is denoted by $\psi(\{x,y\})$ or $\psi(x,y)$. Thus IIA is equivalent to $\psi(x,y) \subseteq \{x,y\}$ for all x,y in X. With countable X, some pairs may not have a relevant set:

Example 1: For any profile $p \in L(X)^N$ set $x \succ_{f(p)} y$ if $x \succ_1^p y$ unless individual 2 has both (i) $y \succ_2^p x$ and (ii) has infinitely many alternatives between y and x in p(2). It is easy to confirm that f is transitive-valued and satisfies Pareto, neutrality, and non-dictatorship. It is also easy to check that $Y \subset X$ is sufficient for a pair {x, y} if and only X\Y is finite. Therefore, there is no relevant set for any pair: If Y is sufficient for {x, y} then so is Y\{y} for any $y \in Y$.

Arrow has shown [1963] that for $|X| \geq 3$ there does not exist any transitive-valued social welfare function satisfying full domain, the Pareto condition, nondictatorship, and IIA. But if we simply delete the requirement of IIA, there are many rules satisfying the rest of Arrow's conditions plus the interprofile conditions (see Fishburn [1987]) of *neutrality* and *anonymity* which we define next.

Suppose that σ is a permutation of N. Such a permutation induces a map σ on profiles where σ(p) assigns ordering p(σ(i)) to individual i. A rule f is ***anonymous*** if for every permutation σ on N and for every profile p in the domain of f, σ(p) is also in the domain and f(σ(p)) = f(p).

Turn now from individuals to alternatives. Any permutation μ of X, the set of alternatives, induces a permutation on preference orders where μ(R) is defined by

$$\mu(x)\mu(R)\mu(y) \text{ if and only if } xRy.$$

In turn, this induces a permutation on profiles where μ(p) assigns ordering μ(p(i)) to individual i

A rule f is ***neutral*** if for every profile p in the domain of f and every permutation μ on X, μ(p) is also in the domain and f(μ(p)) = μ(f(p)).

We clarify these symmetry conditions by contrasting them with other possible versions. Several authors, e.g. Sen [1970, p. 72] and Fishburn [1973, p. 161], define neutrality in such a way as to incorporate considerable independence. An informal version of this is given by Rae and Schickler [1997, p. 167]:

> *Neutrality:* Suppose that all individual ordinal preferences over (x,y) are the same as they are over (w,z), then the collective outcomes over the two pairs of options must be the same.

Because we want to work in weak independence contexts, we do not use their definition.

It is also helpful to contrast our definitions with conditional versions from Campbell and Fishburn [1980]:

Conditional anonymity: For every permutation σ on N and for every profile p in the domain ℘ of f, if σ(p) is also in ℘ then f(σ(p)) = f(p);

Conditional neutrality: For every profile p in the domain ℘ of f and every permutation μ on X, if μ(p) is also in ℘ then f(μ(p)) = μ(f(p)).

Our (unconditional) anonymity requires that ℘ be closed under permutations of individuals; (unconditional) neutrality requires that ℘ be closed under permutations of alternatives.

An impossibility result for infinite X does not follow immediately from an impossibility theorem for the finite case. Consider the following result from Campbell and Kelly (2006) [Theorem 3 there]:

If X is finite with |X| ≥ 3, there does not exist a social welfare function satisfying full domain, transitivity, Pareto, nondictatorship, weakest independence, and neutrality.

Suppose X is infinite and there exists a social welfare function f on X satisfying full domain, transitive-valuedness, Pareto, nondictatorship, weakest independence, and neutrality. Pick a finite subset Y of X with |Y| ≥ 3 and select an ordering Q on X\Y. Define g on Y as follows, for each profile q on Y, extend each q(i) to all of X by appending Q below q(i) to create p(i). Then g(q) is defined to be f(p)|Y. This g function inherits from f the full domain condition, Pareto, and neutrality. But it need not inherit either nondictatorship or weakest independence (see Example 5 below). So we cannot use the nonexistence of a rule satisfying full domain, transitive-valuedness, Pareto, nondictatorship, weakest independence, and neutrality for finite X to rule out the existence of such a rule when X is countably infinite.

Section 2. Examples.

For finite X, a paradigm example of a neutral and anonymous rule satisfying all of Arrow's conditions except IIA is Borda's rule. Borda's rule violates even weakest independence. As commonly defined this rule does not work in the countably infinite context as there need not be a "first" (topmost) element, or "second," etc. However, there is an alternative definition that works for both finite and countably infinite X (with a somewhat restricted domain). The resulting rule satisfies transitive-valuedness, Pareto, non-dictatorship, anonymity, and neutrality (but violates both full domain and weakest independence):

Example 2: Let X be countable and let S be the subset of L(X) such that for every ordering in S and every x and y in X, there are at most finitely many alternatives between x and y. Then the following is a transitive-valued, Paretian, neutral and anonymous social welfare function on $\wp = S^N$. At profile p, for any pair x,y of distinct alternatives in X, let A(i,x,y) be defined as follows:

(1) if $x \succ_i y$, then A(i,x,y) is (1 + the number of alternatives between x and y in $\succ_i$);

(2) if $y \succ_i x$, then A(i,x,y) is the <u>negative</u> of (1 + the number of alternatives between x and y in $\succ_i$).

Then $x \succ_{f(p)} y$ if and only if $\sum_{i\in N} A(i,x,y) > 0$.

This example can be extended to all of $L(X)^N$ so as to still satisfy neutrality (but not anonymity):

Example 3: Let S be as in Example 2. For p in S^N, let f(p) also be determined as in Example 2. If p is in $L(X)^N \backslash S^N$, let i be the individual with the lowest label such that $p(i) \notin S$ and set f(p) = p(i). Because S is closed under permutations of alternatives, this rule is neutral.

Hence, even with countably infinite X, there do exist neutral rules that satisfy all of Arrow's conditions except IIA. We'll see in the next section how far we will have to deviate from IIA.

Turning from neutrality to anonymity, if X is countably infinite, there <u>do</u> exist social welfare functions satisfying full domain, transitivity, the Pareto criterion, non-dictatorship, and anonymity. Many examples below are, like Example 4, weighted-scoring rules that employ "utility" representations. We are, of course, aware of the problems presented <u>in some contexts</u> by utilitarianism.

Example 4: Since X is countable we can write it as a list: X = <x(1), x(2), ... >. Let $\succ$ be an arbitrary strong order on X; there is a numerical representation of $\succ$ (in fact, one where all images are rational), i.e., there is a rational-valued function u on X such that $u(x) > u(y)$ just when $x \succ y$. See Birkhoff [1940 p. 200]; also Fishburn [1970],and Rader [1972]. For each $\succ$ select one such representation. Now, given a profile r = (r(1), r(2), . . . , r(n)) in $L(X)^n$, let u_i be the chosen representation of r(i) and set $x \succeq y$ in the social ranking just when

$$\sum_{i=1}^{n} u_i(x) \geq \sum_{i=1}^{n} u_i(y).$$

This rule is defined on all of $L(X)^n$ and satisfies Pareto and anonymity. However, other properties of the rule depend on the choice of representations. This is true for neutrality and for independence conditions. For some choices, the rule will satisfy a strong form of independence (relevant sets are all finite); for other choices, X is the relevant set for every pair, in which case we might say that the rule is nowhere independent. To illustrate, we present a rule where no finite set is sufficient for any pair {x,y}.

Example 5: Start with any set of representations; they may, for example, yield small relevant sets. We use these representations u to define a new representation v on L(X). Partition L(X) into $L_1 \cup L_2$, where L_1 consists of all those orderings with a minimal element and $L_2 = L(X)\backslash L_1$. Given $\succ$ in L(X), define v by:

$$v(x(i)) = \begin{cases} u(x(i)) \text{ if } \succ \in L_1 \\ 2u(x(i)) \text{ if } \succ \in L(X)\backslash L_1 \end{cases}$$

The rule is given by setting $x \succeq y$ in the social ranking just when

$$\sum_{i=1}^{n} v_i(x) \geq \sum_{i=1}^{n} v_i(y)$$

Then no finite subset Y of X that contains {x,y} is a sufficient set for {x,y} because the ordering on any finite subset of X cannot be used to determine if $\succ$ is in L_1 or L_2.

In the next example, representations are chosen so that all relevant sets are finite.

Example 6: Since X is countable we can write it as a list: $X = <x(1), x(2), \ldots>$. Let $\succ$ be an arbitrary strong order on X. We will describe a particular numerical representation of $\succ$. Let $u(x(1)) = 1$. If $x(2) \succ x(1)$, assign $u(x(2)) = 2$; if $x(1) \succ x(2)$, assign $u(x(2)) = 0$. Proceeding inductively, suppose that we have defined $u(x(i))$ for all $i < n$. Then $u(x(n))$ is specified by one of the following three statements:

1. If $x(n) \succ x(i)$ for all $i < n$, let θ be given by $u(x(\theta)) = \text{Max}\{u(x(1)), u(x(2)), \ldots, u(x(n-1))\}$, then $u(x(n)) = u(x(\theta)) + 1$.

2. If $x(i) \succ x(n)$ for all $i < n$, let θ be given by $u(x(\theta)) = \text{Min}\{u(x(1)), u(x(2)), \ldots, u(x(n-1))\}$, then $u(x(n)) = u(x(\theta)) - 1$.

3. Otherwise, consider $\succ$ restricted to $\{x(1), x(2), \ldots, x(n)\}$; let x(s) and x(p) be the immediate successor and predecessor respectively of x(n); i.e., $x(s) \succ x(n) \succ x(p)$ and there are no alternatives in $\{u(x(1)), u(x(2)), \ldots, u(x(n-1))\}$ ranked between x(s) and x(p).

Then set

$$u(x(n)) = ½\,[u(x(s)) + u(x(p))].$$

Now, given a profile $r = (r(1), r(2), \ldots, r(n))$ in $L(X)^n$, let u_i be the above representation of r(i) and set $x \succeq y$ in the social ranking just when

$$\sum_{i=1}^{n} u_i(x) \geq \sum_{i=1}^{n} u_i(y)$$

This rule is defined on all of $L(X)^n$, satisfies Pareto and anonymity, and for every pair of alternatives, the relevant set is finite: given x(i) and x(j) in X, with $i > j$, the relevant set for this pair is $\{x(1), x(2), ..., x(i)\}$. If every pair in X has a finite sufficient set , we say the rule satisfies ***finite dependence***.

Section 3. An impossibility result

Now that we know there is a rule, defined on all of $L(X)^n$, that satisfies Pareto, anonymity, and satisfies finite dependence we ask what happens when we substitute neutrality for anonymity. We will show that no such rules exist, for any infinite set of alternatives - including the countable case and whether or not we impose transitivity of social preference or the Pareto criterion. Recall that for x,y in X and a given rule f, $\psi(\{x,y\})$ denotes the smallest sufficient set for {x,y}, <u>if there is such a set</u>.

Theorem. For X infinite, there does not exist a social welfare function on $L(X)^n$ that is neutral but where there is even a single pair {x,y} such that

(1) $\psi(\{x,y\})\backslash\{x,y\} \neq \varnothing$;

(2) $\psi(\{x,y\})$ is a <u>proper</u> subset of X.

Proof: Suppose for social welfare function f on $L(X)^n$ there does exist a pair {x,y} such that for some a and z,

$a \in \psi(\{x,y\})\backslash\{x,y\}$; and

$z \in X\backslash\psi(\{x,y\})$.

Since $a \in \psi(\{x,y\})$, there exist profiles u and u* such that

$u|\psi(\{x,y\})\backslash\{a\} = u^*|\psi(\{x,y\})\backslash a$

with $x \succeq_{f(u)} y$ but $y \succ_{f(u^*)} x$. We construct a profile u′ from u by taking each u(i) and moving only z to the "same position" alternative a has in u*(i).

To be more explicit,

(1) If $u(i)|\psi(\{x,y\}) = u^*(i)|\psi(\{x,y\})$, insert z just above a.

(2) If $u(i)|\psi(\{x,y\}) \neq u^*(i)|\psi(\{x,y\})$, insert z according to the following rules:

A. $u'(i)|X\backslash\{z\} = u(i)|X\backslash\{z\}$; and

B. $(z,w) \in u'(i)$ for all $w \in X\backslash\{z\}$ such that $(a,w) \in u^*(i)$ while

$(w,z) \in u'(i)$ for all $w \in X\backslash\{z\}$ such that $(w,a) \in u^*(i)$;

C. Since $u(i)|\psi(\{x,y\}) \neq u^*(i)|\psi(\{x,y\})$, either in u(i) alternative a is preferred to some w with $(w,a) \in u^*(i)$ — in which case $(a,z) \in u'(i)$ — or, in u(i) alternative a is below some w with $(a,w) \in u^*(i)$ — in which case $(z,a) \in u'(i)$. The resulting $u'(i)$ relation is transitive. Since u and u' agree on $\psi(\{x,y\})$, we have $x \succeq_{f(u')} y$.

Finally, construct profile u'' from u' by interchanging a and z. If f satisfied neutrality, the social ranking at u'' would be obtained from the social ranking at u' by just interchanging a and z. In particular, $x \succeq_{f(u'')} y$. But that contradicts

$$u(i)''|\psi(\{x,y\}) = u^*(i)|\psi(\{x,y\})$$

and $y \succ_{f(u^*)} x$. Therefore f must not satisfy neutrality. □

Note that the proof doesn't use anything close to a full domain. The proof is valid for any domain that is closed with respect to the operations employed to switch alternatives' positions.

Neutrality is one of the virtues of the Borda rule. A serious liability of that rule is that a pair of alternatives typically cannot be socially ordered without obtaining information about each individual's preference relation over the entire set X. The theorem tells us that this drawback applies generally to neutral rules. Given <u>any</u> neutral rule on a full domain, for every pair {x,y} that <u>has</u> a relevant set — i.e., a smallest sufficient set — either $\psi(\{x,y\}) = \{x,y\}$ or X is its only sufficient set. Pairs without relevant sets have as sufficient sets only infinitely large sets; in fact, there must be infinitely many infinitely large sufficient sets. All neutral rules that violate IIA have extremely demanding information processing requirements.

A rule exhibits ***finite neutrality*** if for every profile p in the domain and every permutation μ on X that moves only finitely many alternatives, $\mu(p)$ is also in the domain and $f(\mu(p)) = \mu(f(p))$. Much of what we intend regarding equal treatment of alternatives is captured by finite neutrality.

There does exist a way of choosing utility representations that <u>can</u> lead to finite neutrality.

Example 7: Let ~ be the relation on L(X) that, for arbitrary orderings R and Q in L(X), sets R ~ Q just when $Q = \sigma(R)$ for some permutation σ on X that moves only finitely many alternatives of X. This is an equivalence relation that induces a partition on L(X). For each partition component, select one ordering, R, from the component and then select the representation u(R) given by Example 6. For any Q in the same partition component, there is a unique σ with $Q = \sigma(R)$. Permute the representation in the same way to arrive at the utility representation of Q and then set $x \succeq y$ in the social ranking just when

$$\sum_{i=1}^{n} u_i(x) \geq \sum_{i=1}^{n} u_i(y)$$

This rule satisfies Pareto, anonymity, and finite neutrality on $L(X)^N$. Unfortunately it violates finite dependence, and the need to elicit preference information over an infinite set may make information processing prohibitively expensive.

Example 7 can not be extended to include all infinite permutations. Given a Q in the same partition component as R, there may be many different σs with Q = σ(R). For example, let R be an ordering with x and y on top and then a "double-ended" ordering below:

$$x\ y\ \ldots\ a_3\ a_2\ a_1\ b_1\ b_2\ b_3\ \ldots.$$

Next let Q reverse x and y:

$$y\ x\ \ldots\ a_3\ a_2\ a_1\ b_1\ b_2\ b_3\ \ldots.$$

Q is in the same component of R, but there are infinitely many permutations mapping R to Q. Besides the obvious transposition (x y), there is transposition-plus-a-shift:

$$\begin{array}{cccccccccc} x & y & \ldots & a_3 & a_2 & a_1 & b_1 & b_2 & b_3 & \ldots. \\ \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ y & x & \ldots & a_4 & a_3 & a_2 & a_1 & b_1 & b_2 & \ldots. \end{array}$$

Accordingly, choosing a representation for R does not induce a unique representation for Q.

Finally, we observe that for countable X, there is a fundamental design problem in implementing rules with countably many alternatives. How does an individual submit her preference ordering to the social choice procedure? Obviously it is not possible to write down an infinite order on a ballot. Of course, some orderings have a finite description; the preference

$$\ldots x(6) \succ x(4) \succ x(2) \succ \ldots \succ x(5) \succ x(3) \succ x(1)$$

could be: "I prefer all even numbered alternatives to all odd-numbered alternatives; within each of those two groups I prefer higher-numbered alternatives to lower-numbered alternatives."

The problem with this is that while there are only countably many orderings with finite descriptions, the set L(X) is uncountable. So there are orderings with no finite description and how are such orderings to be submitted?

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