ORGAN TRANSPLANTS, HIRING COMMITTEES, AND EARLY ROUNDS OF THE KAPPELL PIANO COMPETITION

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Abstract

Function g selects exactly k alternatives as a function of the preferences of n individuals. It cannot be manipulated by any individual, assuming that an individual prefers set A to B whenever A can be obtained from B by eliminating some alternatives and replacing each with a preferred alternative. Then there is someone whose k top-ranked alternatives are always selected if: **(i)**. k = 2 and n $ 2; or **(ii)**. k = 3 and n = 2; or **(iii)**. k > 3, n = 2, and g has a unanimity property; or **(iv)**. k > 2, n $ 2, g has a unanimity property, and no coalition can manipulate.

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1. Introduction

There is an important family of problems for which a group must choose a fixed number of alternatives, as a function of the preferences of the individual group members. For instance:

- A university department has two vacant positions, and the choice will be made from the set of candidates as a function of the preferences of the members of the hiring committee. The department cannot put a job opening in the bank for another year.
- Exactly k organs are available for transplant, and a hospital committee has to decide which of the many waiting recipients will have transplant surgery.
- A funding agency will choose k research grant recipients from a large applicant pool.

Of course, there are also two-stage collective decision problems for which k alternatives are selected in the first round, and then in round two a final selection, of a single alternative, is made from these k. Because strategic considerations often guide preference revelation in the first round, the analysis of a social choice function that selects exactly k alternatives may inform the design of the overall procedure.

- A panel of judges selects k contestants to advance to the next round of a piano competition. (See Horowitz, 1990.)

- The American Psychological Association selects a president by using the single transferrable vote procedure to narrow the set of candidates to five. The final selection is made from these five.

An alternative treatment would allow k to depend on the reported preferences, as when a selection committee allocates a given amount of money to scholars submitting research grant applications. This covers a different set of applications than the assumption that k is fixed in advance, the case that we investigate in this paper.

In the case of the transplant example, it would be wrong to assume that a physician on the selection committee has as her goal the maximization of selfish individual utility. But even when the dominant motivation is concern for the welfare of others, and when physicians wish to select a set of transplant recipients according to what is just and fair, there is scope for manipulation. Each jury member wants her own view to prevail. Because there is more than one dimension that can enter a fairness evaluation — the age of the applicant, the urgency of the need for a new organ, and the degree to which the applicant's family is depending on him, etc. — the committee members will have different rankings of the applicants in terms of fairness and justice. Moreover, there will be a domain of conceivable rankings, each of which is plausible. If a member of the selection committee feels strongly about the virtue of her own ranking then she might well report the ranking that precipitates the selection of the set of transplant recipients that better reflects her true preferences than truthful revelation. If that happens the outcome will be less than ideal, assuming that g is designed to employ community norms and standards to select an outcome as a function of the *reported* preferences.

We are concerned about strategic misrepresentation of preference, regardless of the degree to which an individual's preference reflects selfish considerations as opposed to a concern for the welfare of others. The temptation to manipulate may be stronger when the decision maker is reporting preferences for outcomes that more directly concern his own welfare. However, even when the benefit to a committee member from obtaining an outcome that ranks higher in his personal ordering of alternatives is indirect, as in the case of the transplant example, we may need assurance of the absence of even an indirect benefit from manipulation when the process results in outcomes that have such enormous effects on the welfare of those concerned. This is obvious in the case of transplant recipients. In the case of a panel of judges choosing the best pianists, the effect on career earnings can be immense. (See Frank and Cook, 1995).

If exactly two alternatives must be chosen, then we will show that the only rules that never give an individual the incentive to manipulate will ignore the preferences of all but one individual. In fact, the top two alternatives in the preference ordering reported by that individual will be the two alternatives selected by the rule. For our results for arbitrary k we assume that the social choice function in question has a unanimity property. We have been unable to prove that only one person's preferences can influence the selection without assuming unanimity, a property that can be *derived* from strategy-proofness when only one alternative will be chosen. Nor have we been able to show that our result doesn't hold without some such property. We do prove that if k alternatives are to be selected and no *coalition* can ever profit by deviating from truthful revelation then the rule must identify one individual and choose the k highest ranked alternatives in that individual's preference ordering. In addition to assuming that the final choice set contains exactly k alternatives, we also require every subset containing exactly k alternatives to be chosen at some profile.

To discuss manipulability and strategy-proofness we must know when an individual would submit a false order to obtain a different chosen set. This means that we have to know how individuals order k-element sets as a function of their preferences on single alternatives. Because we are modeling the selection of exactly k alternatives from X, we could use the individual preferences over the family X_k of k-element subsets of X as the set of primitive individual preference relations. Then the rule g selects a single member g(p) of X_k as a function of the profile p of reported individual preferences over X_k. However, we cannot apply the theorem of Gibbard (1973) and Satterthwaite (1975) because we do *not* assume that any complete and transitive ordering of X_k is admissible. If k = 2 and X has, say, 101 alternatives then an ordering for which the top 50 pairs contain alternative x and the bottom 50 pairs also contain x might be bizarre for some interpretations of our model. Similarly, it may not be necessary to have a social choice rule that works when someone ranks {x, y} at the top, {x, z} second, and {y, z} last if a priori information tells us that an individual would never have such a preference scheme. Moreover, although the proofs of "impossibility theorems" often only use a fraction of the full domain, there appears to be no ready conversion of an existing proof of Gibbard-Satterthwaite to our problem.

There is another compelling reason to reject the supposition that individuals will be expected to submit a complete ranking of all of the k-element subsets of X. It is implausible that individuals will be asked to submit enough information to allow, say the almost four million four-element subsets of a set X with 101 members to be completely ordered. It is much more reasonable to ask someone to order the members of X. Therefore, the primitive individual ordering is the one on the set X from which the k-element subsets are drawn. Accordingly, we restrict the individual preferences on X_k by assuming that each individual has an *extension principle* that extends her ordering on X to an ordering on $X_{k.}$ The extension principle gives us a systematic way of thinking about the appropriate restrictions on individual preferences over subsets of X.

Pattanaik (1973, 1974), Gärdenfors (1976), Kelly (1977), and Barberà (1977) began the investigation of strategy-proofness of non-resolute social choice rules. Baigent (1998), Duggan and Schwartz (2000), Barberà, Dutta, Sen (2001), Ching and Zhou (2002), and Campbell and Kelly (2000, 2002) have resumed the inquiry. Each paper employs a different assumption about the way that individual preference over sets is generated from the primitive individual preference over alternatives, but all of them assume that every profile of linear orderings of the singleton alternatives in X is admissible. The result in each case is essentially an impossibility theorem. Even though g(p) contains more than one alternative, Ching and Zhou (2002), Barberà, Dutta, and Sen (2001), and Duggan and Schwartz (2000) assume that a single alternative will eventually be chosen from g(p) by some random process. Therefore, they ground their extension principles in expected utility.

This paper is concerned with situations in which exactly k alternatives will ultimately be selected. Campbell and Kelly (2002) address this question (among others), but without assuming that the rule cannot be manipulated by coalitions of two or more members, but *with* the assumption that everyone employs leximin extension, a complete ordering of X_k, and then prove that a strategy-proof rule must be dictatorial if it selects a fixed number of alternatives at each profile. An extension principle that yields a linear order on the family of k-element sets appears to be far too demanding, especially for situations in which the committee member's welfare is only indirectly affected by the outcome. This paper employs a fairly weak extension property. We merely assume that an individual will manipulate to get the subset Y instead of Z if it is possible to transform Z into Y by replacing some members of Z with preferred alternatives. We say that an extension rule has the *replacement property* in that case.

2. Definitions and Notation

We begin by fixing a positive integer $k > 1$. The finite set X of alternatives has more than k members. We let X_k denote the collection of all k-element subsets of X. For any $Y \subset X$ and any binary relation R we let $R|Y$ denote the relation $R \cap Y \times Y$, the restriction of R to Y. A *linear ordering* $\succ$ on X is a transitive relation on X such that either $x \succ y$ or $y \succ x$ (but not both) holds for any distinct x and y in X. L(X) denotes the set of linear orderings on X. We write $x \succeq y$ if either $x = y$ or $x \succ y$.

A *profile* r = (r(1), r(2), ..., r(n)) is a function from the set N = {1, 2, ..., n} into L(X). We use $x \succ_i^r y$ to denote the fact that individual i strictly prefers x to y at profile r. For $Y \subset X$ and any profile r, we let $r|Y$ denote $(r(1)|Y, r(2)|Y, ..., r(n)|Y)$, the restriction of r to Y. If $r(i) \in L(X)$ then we designate the top-most element by r(i)[1], the second-ranked element by r(i)[2], the *set* containing the top two by r(i)[1,2], etc. We abbreviate r(i)[1,2,..., k] by r(i)[1:k]. (When convenient we treat r(i)[j] as a set, and sometimes we treat it as a member of X.)

For the purpose of this paper, a *social choice rule* g is a map from $L(X)^N$ into X_k. Profile p is an *i-variant* of profile r if p differs from r only in its value for individual i. For $k > 3$ and $n = 2$ our results apply only to social choice rules that select k members of r(1)[1:k+1] whenever r(1)[1:k+1] = r(2)[1:k+1]. The *range* of g is the set of Y in X_k such that g(p) = Y for some profile p in the domain of g.

Extended unanimity If n = 2 then $g(r) \subset r(1)[1:k+1]$ if $r \in L(X)^N$ and $r(1)[1:k+1] = r(2)[1:k+1]$.

For k = 2 and 3 we are able to derive *extended unanimity* from strategy-proofness.

An *extension principle* E maps each ordering $\succ$ in W(X) into a binary relation $E(\succ)$ on X_k. If at profile p the subset Y ranks strictly above Z in the extension of p(i), and it's clear which extension principle is employed, we will typically say that "p(i) ranks Y above Z." We will assume that each individual's extension principle has the *replacement property*, which means that set Y ranks strictly above set Z if Y can be obtained from Z by replacing some members of Z with strictly preferred alternatives:

The replacement property: If Y and Z are two sets of cardinality k then an individual with the preference ordering $\succ$ on X prefers Y to Z under an extension principle with the replacement property if we can number the elements of the sets $Y = \{y_1, y_2,...,y_k\}$ and $Z = \{z_1, z_2,...,z_k\}$ such that $y_t \succeq z_t$ for all $t \in \{1,2...,k\}$, and $y_t \succ z_t$ for some t. (Note that if k = 1 then $\{x\} \succ \{y\}$ if and only if $x \succ y$.)

The following extension principle clearly satisfies the replacement property:

The substitution extension principle. Y is preferred to Z if and *only if* we can number the elements of the sets $Y = \{y_1, y_2,...,y_k\}$ and $Z = \{z_1, z_2,...,z_k\}$ such that $y_t \succeq z_t$ for all $t \in \{1,2...,k\}$, and $y_t \succ z_t$ for some t.

Any other extension satisfying the replacement property will rank Y above Z for a given p(i) if substitution extension ranks Y above Z at p(i). It is obvious that if X has k+1 members, then substitution gives a complete ordering of the members of X_k for any ordering in L(X). However, substitution extension will not give a complete ranking of X_k if X has more than k+1 members: If the primitive ordering $\succ$ of X has $w \succ x \succ y \succ z$ then the substitution extension of $\succ$ does not have {w,z} ranking above {x,y}, nor does it have {x,y} ranking above {w,z}. It is clearly far from complete.

Note that the substitution extension of any weak ordering is asymmetric, as a consequence of the transitivity of the relation $\succ$ on X. The following obvious property of substitution extension will be used in the next section without explicitly citing it.

Lemma 1: The substitution extension of an arbitrary *linear* ordering $\succ$ in L(X) ranks the set consisting of the top k elements of $\succ$ above every other k-element subset of X. Similarly, for any large enough subset X' of X, the set consisting of the top k elements of the ordering $\succ|X'$ ranks above every other k-element subset of X'.

Proof: Let $Y = \{x_1, x_2, ..., x_k\}$ be the set consisting of the k top ranked alternatives from X according to $\succ$. Assume that the alternatives are numbered so that $x_1 \succ x_2 \succ ... \succ x_{k-1} \succ x_k$. Let $Z = \{z_1, z_2, ..., z_k\}$ be some other k-element set, numbered so that $z_1 \succ z_2 \succ ... \succ z_k$. Let j be the smallest integer such that $x_j \notin Z$. Then $x_j \succ z_j$ and $x_i = z_i$ for all $i < j$. And we have

$$x_{j+1} \succ x_{j+2} \succeq z_{j+1}, \text{ and } x_{j+2} \succ x_{j+3} \succeq z_{j+2}$$

and so on. We get $x_i \succ z_i$ for all $i \geq j$. $\square$

As usual an individual manipulates when he reports a false preference ordering in order to precipitate an outcome that he prefers, according to his true preference scheme, to the one that emerges when that true preference is submitted. However, when more than one alternative is selected, the formal definition depends on the extension principle employed.

Manipulation: Social choice rule g is manipulable at profile p by individual i, if there exists an i-variant r of p such that g(r) is strictly preferred to g(p) according to the extension of p(i). And g is manipulable at p by *coalition* $J \subseteq N$ if there are two profiles p and r such that p(i) = r(i) for all $i \in N \backslash J$ and each $j \in J$ prefers g(r) to g(p) under the extension of p(j).

We will identify a set of modest conditions that imply that the social choice rule is dictatorial, assuming that individuals employ substitution extension. It follows that same set of conditions imply that the social choice rule is dictatorial if any individual employs any extension principle with the replacement property. Therefore, we will merely assume substitution extension from now on.

Strategy-proofness: Social choice rule g is strategy-proof if no *individual* can manipulate at any profile.

Dictatorship: If $\wp \subseteq L(X)^N$ then person i is a dictator for $g: \wp \to X_k$ if for every admissible profile $r \in \wp$ the set g(r) ranks higher in the extension of r(i) than any other member of the range of g. In general, for $\wp \subseteq W(X)^N$, person i is a dictator for $g: \wp \to X_k$ if for every admissible profile $r \in \wp$ there is no member of the range of g that ranks higher in the extension of r(i) than g(r).

The following three examples provide some insight into our dictatorship result for the selection of exactly two alternatives.

Example 1: Choose person 1's top ranked alternative and person 2's top-ranked alternative if they are different. If they are the same chose the two alternatives in person 1's ordering. If the preferences are as

p(1)	p(2)
x	x
z	y
y	z
⋮	⋮

in the table then, because the substitution extension of p(2) ranks {x, y} above {x, z}, person 2 can manipulate at p because g(p) = {x, z}, but the set {x, y} will be chosen if person 2 reports an ordering with y on top.

Example 2: There are three individuals and X = {x, y, z}. Choose the majority winner if there is one, and also the majority winner from X\{w} where w is the overall majority winner. If there is no overall majority winner select the majority winner from {x, y} and also the majority winner from X\{w} where w is the majority winner from {x, y}. If the preferences are as in the table then person 3 can manipulate

p(1)	p(2)	p(3)
x	y	z
y	z	x
z	x	y

at p because g(p) = {x, y} but {y, z} will be selected if person 3 reports the same ordering as individual 2.Note that the substitution extension of p(3) ranks {y, z} above {x, y}.

Example 3: Select whichever of person 1's top two alternatives is preferred by person 2, and also select the alternative preferred by person 2 in the set consisting of person 1's third ranked alternative and whichever alternative from 1's top two was not selected initially. If the preferences are as in the table then person 1 can manipulate at p because g(p) = {x, z}, although {x, y} will be selected if person 1 reports (x, w, y, z, ⋯), the ordering with x first, then w, then y, then x. Note that the substitution extension of p(1) ranks {x, y} above {x, z}.

p(1)	p(2)
x	x
y	z
z	y
w	w
⋮	⋮

Because we are mainly concerned with manipulation by a single individual, when we compare the output of a social choice rule at two different profiles p and r we will have to do so by tracking the output on a sequence of profiles beginning at p and ending at r and which changes only one individual preference at each step.

Standard sequence: Given two profiles p and r, the standard sequence from p to r is the following sequence $\{p^t\}$ of profiles

$$p = p^0 = (p(1), p(2), p(3), ..., p(n))$$

$$p^1 = (r(1), p(2), p(3), ..., p(n))$$

$$p^2 = (r(1), r(2), p(3), ..., p(n)$$

$$\vdots$$

$$p^n = (r(1), r(2), r(3), ..., r(n)) = r.$$

Note that successive terms p^{t-1} and p^t of the standard sequence are t-variants, because

$$p^{t-1} = (r(1), r(2), ..., r(t-1), p(t), p(t+1), ..., p(n)), \text{ and}$$

$$p^t = (r(1), r(2), ..., r(t-1), r(t), p(t+1), ..., p(n)).$$

3. The results

We assume throughout that X has more than k members and that each person employs the substitution extension principle. We prove that if the rule $g: L(X)^N \to X_k$ has range X_k and is invulnerable to manipulation then it is dictatorial. We first prove this claim for n = 2, and when invulnerability to manipulation means that no *individual* can manipulate g. Then we prove the claim for any n, but to do so we have to assume that g cannot be manipulated by any coalition. Finally, we prove the claim for arbitrary n and k = 2, but without having to assume that g cannot be manipulated by any coalition of two or more persons. Because we employ the substitution extension principle, each result is valid for any extension principle with the replacement property. We begin with a simple and basic lemma. It is valid even when X contains only two members and k = 1.

Lemma 2: Suppose that $g{:}L(X)^N \to X_k$ is strategy-proof under substitution extension and $Y \in X_k$ belongs to the range of g. If $p(i)[1{:}k] = Y$ for all $i \in N$ then $g(p) = Y$.

Proof: We have $g(q) = Y$ for some q by the range assumption. If $p(i)[1{:}k] = Y$ for all $i \in N$ then let $\{q^t\}$ be the standard sequence from q to p. We know that $g(q^0) = Y$. Suppose $g(q^t) = Y$. If $g(q^{t+1}) \neq Y$ then t+1 can manipulate g at q^{t+1} because the substitution extension of $q^{t+1}(t+1) = p(t+1)$ ranks Y above every other k-element subset of X. Therefore, $g(q^{t+1}) = Y$. Then $g(p) = g(q^n) = Y$ by induction. □

The starting point for all of our strategy-proofness-implies-dictatorship results is the case n = 2. Lemma 2 implies that when n = 2, for any profile r, arbitrary individual i has the power to ensure that the outcome is $r(j)[1{:}k]$ for $j \neq i$. That follows from the fact that $g(r') = r(j)[1{:}k]$ when $r'(j) = r(j) = r'(i)$. It is often said that $r(j)[1{:}k]$ belongs to person i's *option set* at r (Barberà, 1983). However, we prefer to say that "person i can get $r(j)[1{:}k]$" leaving the reference to Lemma 2 implicit.

Our first theorem applies only to a two-person group. If k = 2 or 3 then strategy-proofness and the range assumption implies extended unanimity. We suspect that this is the case for any k, but we have not been able to prove that — or come up with a counterexample. Hence we assume that g has the extended unanimity property if k is greater than 3.

***Theorem 1*:** Assume that X contains at least k+1 alternatives and n = 2. Suppose that $g{:}L(X)^N \to X_k$ is strategy-proof under substitution extension, the range of g is X_k, and g satisfies extended unanimity if $k \geq 4$. Then some individual is a dictator for g.

Given a (k+1)-element subset Z of X and a fixed profile $q \in L(X)^N$ let $\wp^Z$ be the set of profiles p $\in \wp$ such that $p(i)[1{:}k+1] = Z$ for i = 1 and 2 and $p|X\backslash Z = q|X\backslash Z$. For arbitrary $p \in \wp^Z$, we set $g^Z(p) = g(p)$, to define g^Z. If k > 3, extended unanimity gives us $g^Z(p) \subset Z$ for any $p \in \wp^Z$. For k = 2 or 3 we need another lemma before we turn to the proof of the proposition. When we say that person i's top belongs to g(p), we mean that $p(i)[1] \in g(p)$.

Lemma 3. Assume that k = 2 or 3 and n = 2. Let Z be any subset of X with exactly k+1 members.

If g is strategy-proof under substitution extension, and the range of g is X_k, then $g^Z(p)$ is a subset of Z.

Proof: **Step 1**: We begin by showing that strategy-proofness of g implies that $g^Z(p)$ is a subset of Z if k = 2, and also if k = 3 and p(1)[1] = p(2)[1]. Profile p is in $\wp^Z$, so p(i)[1:k+1] = Z and $p(i)|X\backslash Z = q(i)|X\backslash Z$ for i = 1 and 2. Suppose that g(p) is *not* a subset of Z. That is, there is some alternative a in g(p) but not in p(1)[1:k+1] = p(2)[1:k+1]. Because person i can get p(j)[1:k] for j ≠ i we must have p(i)[1] ∈ g(p) for i = 1, 2. That follows from the fact that p(i) ranks p(j)[1:k] above every k-element subset of X that contains a but not p(i)[1]. Then a ∈ g(p) ∩ X\Z and k = 2 implies p(1)[1] = p(2)[1] because both tops belong to g(p). But r(2)[2] $\succ^p_1$ a and thus person 1 can manipulate g at p because person 1 can get r(2)[1,2]. Therefore, g(p) ⊂ Z if k = 2.

Suppose a ∈ g(p) ∩ X\Z and k = 3. If p(1)[1] = p(2)[1] then g(r) = {x, a, z} for some z ∈ X\{a,x} and x = p(1)[1]. Then p(1) will prefer p(2)[1:3] to g(p) unless p(1)[2] ∈ g(p) and p(1)[2] ∉ p(2)[1:3]. (If p(1)[2] ∉ g(p) then person 1 can manipulate at p because p(2)[1:4]\p(1)[2] is preferred by p(1) to every other three-element subset of X\p(1)[2]. If p(1)[2] ∈ p(2)[1:3] then obviously p(1) ranks p(2)[1:3] above g(p).) But p(1)[2] ∉ p(2)[1:3] implies p(1)[2] = p(2)[4], in which case p(1)[2] ∈ g(p) implies

g(p) = {p(2)[1], p(2)[4], p(2)[t]} = {p(1)[1], p(1)[2], p(2)[t]} for some t ≥ 5.

Then p(2) ranks p(1)[1:3] above g(p). Therefore, strategy-proofness of g implies $g^Z(p)$ ⊂ Z if p(1)[1] = p(2)[1].

Step 2: The next step is to show that if k = 3 and $g^Z(p)$ is a *not* subset of Z then person 1's top-ranked alternative is person 2's fourth-ranked alternative, and person 2's top-ranked alternative is person 1's fourth-ranked alternative.

Suppose a ∈ g(p) ∩ X\Z, k = 3, and p(1)[1] = x ≠ y = p(2)[1]. Then g(p) = {x, y, a}. (Recall that both tops belong to g(p).) If x ∈ p(2)[1:3] then p(1) ranks p(2)[1:3] above g(p) and if y ∈ p(1)[1:3] then p(2) ranks p(1)[1:3] above g(p). Therefore, strategy-proofness of g implies x = p(2)[4] = p(1)[1] and y = p(1)[4] = p(2)[1].

Step 3: Finally we show that even when person 1's top-ranked alternative is person 2's fourth-ranked alternative, and person 2's top-ranked alternative is person 1's fourth-ranked alternative g(p) must be a subset of Z (assuming k = 3). We have established that if p ∈$\wp^Z$ and g(p) is not a subset of Z then k = 3 and p has the form of the profile of Table 1 for which {c, d} = {w, z}, and g(p) = {x, y, a} for some a ∈

X\Z:

Table 1

p(1)	p(2)
x	y
w	c
z	d
y	x
⋮	⋮
a	a
⋮	⋮

We will use Table 2 to arrive at a contradiction. Profile t of Table 2 belongs to $\wp^Z$.

Table 2

t(1)	t(2)
w	z
z	w
x	y
y	x
⋮	⋮
a	a
⋮	⋮

If $g(t(1),p(2)) \neq Z\backslash\{x\}$ then person 1 can manipulate at p by reporting t(1) because g(t(1),p(2)) is a subset of Z by Step 2 and g(t(1),p(2)) will contain x and two other members of Z, one of which will rank at least as high as y in p(1) and the other will rank higher than a in p(1). Similarly, $g(p(1),t(2)) \neq Z\backslash\{y\}$ implies that person 2 can manipulate g at p by reporting t(2). Therefore, we have

$$g(t(1),p(2)) = Z\backslash\{x\} \text{ and } g(p(1),t(2)) = Z\backslash\{y\}.$$

It follows that g(t) = {x, w, z} because person 1 can get t(1)[1:3] by reporting p(1). But we also conclude that g(t) = {y, w, z} because person 2 can get t(2)[1:3] by reporting p(2). The contradiction forces us to conclude that $g(p) \subset Z$ for the profile p of Table 1. For k = 3 we have $g^Z(r) \subset Z$ for all $r \in \wp^Z$.

□

***Proof of Theorem 1*:** We proceed by induction on $\delta(r)$, which we define as the cardinality of the smallest subset Y of X such that $|Y| \geq k$ and, for all $i \in N$, $y \succ_i^r z$ for all $y \in Y$ and all $z \in X\backslash Y$. There is such a smallest set for any profile because we can begin by setting Y = X. We start with $\delta(r) = k + 1$.

Lemmas 2 and 3 and extended unanimity imply that for every (k+1)-element subset Z of X the range of g^Z is $\mathbf{Z_k}$, the set of k-element subsets of Z.

Step 1: We prove that there is a dictator for g^Z for every (k+1)-element subset Z of X. Let $\succ$ be a linear ordering on $Z = \{x_1, x_2, ..., x_k, x_{k+1}\}$, where we have numbered the members of Z so that $x_h \succ x_j$ if and only if $h < j$. Then if $\succ^*$ is the substitution extension of $\succ$ we have $Z\backslash\{x_j\} \succ^* Z\backslash\{x_h\}$ if and only if $h < j$. Note that this is a complete ordering on Z_k. And conversely, if $\succ^*$ is a linear ordering on X_k then $\succ^*$ is the substitution extension of one and only one linear ordering on Z, as we now prove. Number the elements of $\mathbf{Z_k} \equiv \{Z_1, Z_2, ..., Z_k, Z_{k+1}\}$ so that $Z_h \succ^* Z_j$ if and only if $h < j$. And number the elements of $Z \equiv \{z_1, z_2, ..., z_k, z_{k+1}\}$ so that $Z_j = Z\backslash\{z_j\}$. Then $\succ^*$ is the substitution extension of the linear ordering $z_{k+1} \succ z_k \succ ... \succ z_2 \succ z_1$. Therefore, there is an isomorphism between $P(Z)^N$ and the set of profiles of substitution extensions of the members of $P(Z)^N$. We can think of $\mathbf{Z_k}$ as the alternative set for g^Z. With this interpretation in mind, if $\wp = L(X)^N$ then $\wp^Z$ satisfies the domain assumption of the Gibbard-Satterthwaite theorem. We have established that g^Z satisfies the range assumption of the Gibbard-Satterthwaite theorem. Therefore, g^z is dictatorial for every (k+1)-element subset Z of X.

We now show that if person 1 is a dictator for g^Z, and Z is a (k+1)-element subset of X, then we have g(r) = r(1)[1:k] whenever r(1)[1:k+1] = Z = r(2)[1:k+1], whether or not $r|X\backslash Z = q|X\backslash Z$. Assume that r(1)[1:k+1] = r(2)[1:k+1] = Z. If r(1)[1:k] = r(2)[1:k] then Lemma 2 gives us the desired conclusion. Assume then that r(1)[1:k] ≠ r(2)[1:k]. Define p by setting p(1) = r(1) and $p(2)|Z = r(2)|Z$ with $p(2)|X\backslash Z = q(2)|X\backslash Z$. Then strategy-proofness of g implies g(p) = p(1)[1:k] = r(1)[1:k] because person 1 can get p(1)[1:k] by reporting an ordering that ranks the members of X\Z according to q(1). Let g′ be the social choice rule that is defined in the same way as g^Z except that we replace q with (p(1),q(2)). Then g′ must be dictatorial by the previous paragraph, and the dictator must be person 1 because g(p) = p(1)[1:k} ≠ p(2)[1:k]. Now define q″ exactly as we defined g^Z except that we replace q with r. The dictator for q″ must person 1, otherwise person 2 could manipulate at p′ when p′(1) = r(1), $p'(2)|X\backslash Z = q(2)|X\backslash Z$ and p′(2)[1:k] ≠ r(1)[1:k]. Therefore, we have established that g(r) = r(1)[1:k] whenever r(1)[1:k+1} = r(2)[k+1]. Therefore, we can think of g^Z as defined whenever Z has k+1 members and r(1)[1:k+1] =

r(2)[1:k+1]. We have shown that g^Z will be dictatorial.

If X has exactly k+1 members then we are finished with the proof of the proposition. Suppose, then, that X has more than k+1 members. We now show that there is an individual i such that g(p) = r(i)[1:k] whenever r(1)[1:k+2] = r(2)[1:k+2], and hence that we can think of g^Z as defined whenever Z has k+2 members and r(1)[1:k+2] = r(2)[1:k+2]. Consider any k−1 element set $A \subset X$ and three distinct alternatives x, y, and z in X\A. We will show that some person is a dictator for $g^{A\cup\{x,y,z\}}$. Our first step is to show that $A \subset g(r) \subset A \cup \{x,y,z\}$ for any profile $r \in \wp$ such that r(1)[1:k−1] = A = r(2)[1:k−1] and r(1)[1:k+2] = $A \cup \{x,y,z\}$ = r(2)[1:k+2]. We begin by representing r as follows :

Table 3

r(1)	r(2)
A	A
x	r(2)[k]
y	r(2)[k+1]
z	r(2)[k+2]
⋮	⋮

(We do not mean to suggest that r(1) and r(2) order the members of A in the same way, but merely that r(1)[1:k−1] = r(2)[1:k−1]. Note that the three alternatives ranked below A in r(2) are x, y, and z in some order.) If r(2)[k+2] = z then $A \subset g(r)$ by the previous paragraph because some individual is a dictator for $g^{A\cup\{x,y\}}$. Suppose r(2)[k+2] = y. If r(2)[k] = x then g(r) = $A \cup \{x\}$ by Lemma 2. Therefore, if r(2)[k+2] = y we only have to look at the case r(2)[k] = z. Then we have the profile of Table 4:

Table 4

r(1)	r(2)
A	A
x	z
y	x
z	y
⋮	⋮

Person 2 can get $A \cup \{x\}$ by Lemma 2, and r(2) ranks $A \cup \{x\}$ above every other k-element subset of X except $A \cup \{z\}$. Therefore, strategy-proofness implies either $g(r) = A \cup \{x\}$ or $g(r) = A \cup \{z\}$.

We conclude that if A is not a subset of g(r) then we must have $r(2)[k+2] = x$. We can rule out $r(2)[k] = y$ by the previous paragraph, just by interchanging the roles of the two individuals. Therefore, only one case remains:

Table 5

r(1)	r(2)
A	A
x	z
y	y
z	x
⋮	⋮

Person 1 can get $A \cup \{z\}$ which is preferred by r(1) to any member of X_k except sets containing x or y. If A is not a subset of g(r) but g(r) does not contain *both* x and y then person 1 can manipulate g at r, This follows from the fact that r(1) ranks $A \cup \{z\}$ above every k-element subset of $X\backslash\{x\}$ that does not contain A and to every other k-element subset of $X\backslash\{y\}$ that does not contain A. Therefore, both x and y belong to g(r). The same argument from the standpoint of person 2 will show that g(r) must contain z and y. Therefore, if A is not a subset of g(r) then $\{x,y,z\} \subset g(r)$. But then there must be two members of A, call them a and b, that do not belong to g(r). Now, person 1 can get $A \cup \{z\}$, and $a \succ_1^r x$ and $b \succ_1^r y$. Therefore, person 1 can manipulate g at r. We must have $A \subset g(r)$ after all.

We have proved that $A \subset g(r)$ for any profile r such that $r(1)[1{:}k-1] = A = r(2)[1{:}k-1]$ and $r(1)[1{:}k+2] = r(2)[1{:}k+2]$. But this means that $g(r) \subset r(1)[1{:}k+2]$: Person 1 can get $A \cup r(2)[k]$ which r(1) ranks above $A \cup \{w\}$ for any w not belonging to $r(1)[1{:}k+2]$.

A is an arbitrary subset of X with $k-1$ elements and x, y, and z are any three members of $X\backslash A$. Let $\wp_A$ be the set of all $p \in L(X)^N$ such that $p(i)[1{:}k-1] = A$ and $p(i)[1{:}k+2] = A \cup \{x,y,z\}$ for $i = 1,2$. We have $A \subset g(p) \subset A \cup \{x,y,z\}$ for all $p \in \wp_A$. Consider the single-valued social choice rule g_A with domain $\wp_A$ defined by setting $g_A(p) = g(p)\backslash A$. Lemma 2 implies that the range of g_A is $\{x,y,z\}$. Therefore, the Gibbard-Satterthwaite theorem implies that g_A is dictatorial. (Note that g_A is strategy-proof: Suppose $g_A(p') \succ_i^p g_A(p)$ for some p' in $\wp_A$ that is an i-variant of p. But then p(i) ranks $A \cup g_A(p')$ above g(p), contradicting the strategy-proofness of g.) Suppose that person 1 is the dictator for $g^{A \cup \{x,y\}}$. Then if q is

Table 6

q(1)	q(2)
A	A
x	y
y	x
z	z
⋮	⋮

the profile of Table 6 we have $g(q) = A \cup \{x\}$. Then person 1 is the dictator for g_A. This means that g(s) $= A \cup \{x\}$ if s is the profile of Table 7. Therefore, person 1 is the dictator for $g^{A \cup \{x,z\}}$. Similarly,

Table 7

s(1)	s(2)
A	A
x	z
z	x
y	y
⋮	⋮

if person 1 is a dictator for $g^{A \cup \{x,y\}}$ then he is a dictator for $g^{A \cup \{y,z\}}$. Choose any alternative b not in $A \cup \{x,y,z\}$. At the profile p of Table 8 we have $g(p) = A \cup \{x\}$ because person 1 can get $A \cup \{x\}$ by

Table 8	
p(1)	p(2)
A	A
x	y
y	x
b	z
⋮	⋮

reporting q(1) of Table 6. Consider profile u of Table 9. It is the same as p of Table 8 except that

Table 9	
u(1)	u(2)
A	A
x	y
y	x
b	b
⋮	⋮

we replace z in person 2's ordering with b. If $g(u) = A \cup \{y\}$ then person 2 can manipulate at p by reporting u(2). Therefore, $g(u) \neq A \cup \{y\}$. But u(2) ranks $A \cup \{x\}$ above any other k-element subset of X except $A \cup \{y\}$. Therefore, $g(u) = A \cup \{x\}$ by strategy-proofness of g. Therefore, person 1 must be the dictator for $g^{A \cup \{x,b\}}$. By substituting one alternative at a time in this way we can show that some individual i is a dictator for g^Z for every (k+1)-element subset Z of X.

Recall that $\delta(r)$ is the cardinality of the smallest subset Z of X such that every individual ranks every member of Z above every member of X\Z at r (and $|Z| \geq k$). We next show that g(r) = r(1)[1:k] if $\delta(r) = k$ or k+1, assuming that person 1 is the dictator for g^Z.

Step 2 Set $Z = \{x_1, x_2, ..., x_k, x_{k+1}\}$ and let r be the profile:

Table 10

r(1)	r(2)
x_1	x_1
x_2	x_2
⋮	⋮
x_k	x_{k+1}
x_{k+1}	x_k
⋮	⋮

Either person 1 or 2 is a dictator for g^Z. Therefore, g(r) = Z\{x_{k+1}} or Z\{x_k}. Without loss of generality, assume that g(r) = Z\{x_{k+1}}. We will prove that person 1 is a dictator for g. That is, we have to show that g(r) = r(1)[1:k] for all r. If δ(r) = k then g(r) = r(1)[1:k] by Lemma 2. If δ(r) = k+1 then g(r) = r(1)[1:k] by step 1. We have proved that g(r) = r(1)[1:k] for all r such that δ(r) = k or k+1. Of course, the final step is to extend this to all values of δ(r).

Step 3. Suppose that g(r) = r(1)[1:k] for all r such that δ(r) < ℓ and in the present instance we have δ(r) = ℓ. If r(2)[ℓ] ∉ r(1)[1:k] then strategy-proofness implies that g(r) = r(1)[1:k] because the induction hypothesis implies that g(r′) = r′(1)[1:k] for r′ satisfying r′(2) = r(2), r′(1)[1:k] = r(1)[1:k], and r′(1)[ℓ] = r(2)[ℓ]. Now, consider the following statement D(j):

D(j) g(r) = r(1)[1:k} when r(2)[ℓ] = r(1)[j].

We know that D(ℓ) is true. We now prove that D(j) implies D(j−1). D(j−1) is obviously true for j−1 > k, by strategy-proofness, because the induction hypothesis implies that g(r′) = r′(1)[1:k] for r′ satisfying r′(2) = r(2), r′(1)[1:k] = r(1)[1:k], and r′(1)[ℓ] = r(2)[ℓ]. And D(j−1) holds if D(j) is true for j ≤ k because then we have g(r′) = r′(1)[1:k] = r(1)[1:k] if r′(2) = r(2) and r′(1) is the same as r(1) except that r′(1)[j−1] = r(1)[j] and r′(1)[j] = r(1)[j−1]. Therefore, we must have g(r) = r(1)[1:k] by strategy-proofness. It remains to prove that D(k) holds if D(k+1) does. Suppose that D(k+1) holds, r(2)[ℓ] = r(1)[k], but g(r) ≠ r(1)[1:k]. Then person 1 can get r(1)[1:k+1]\r(1)[k] because D(k+1) implies that g(r′) = r′[1:k] if r′(2) = r(2) and r′(1) is the same as r(1) except that r(1)[k] and r(1)[k+1] exchange places. But r(1) ranks r(1)[1:k+1]\r(1)[k] above every k-element subset of X except r(1)[1:k]. Therefore, g(r) ≠ r(1)[1:k] implies g(r) = r(1)[1:k+1]\r(1)[k]. But then person 2 can manipulate g at r″ if we define r″ so that r″(1) = r(1) and r″(2) is the same as r(1) except that r(1)[k] and r(1)[k+1] exchange places: We have

$g(r'') = r''(1)[1{:}k] = r(1)[1{:}k]$ by $D(k+1)$. And $g(r''(1),r(2)) = g(r) = r(1)[1{:}k+1]\backslash r(1)[k]$, and $r''(2)$ ranks that set above $r(1)[1{:}k]$ because $r(1)[k+1]$ ranks above $r(1)[1]$ in $r''(2)$. Therefore, we must have $g(r) = r(1)[1{:}k]$ if g is strategy-proof. Then $D(k)$ holds if $D(k+1)$ does, and thus $D(j)$ holds for all j by induction. This obviously implies that $g(r) = r(1)[1{:}k]$ for all r such that $\delta(r) = \ell$. By induction on ℓ, person 1 is a dictator for $g^X = g$. □

Now we show how Theorem 1 can be used to extend the dictatorship result to $n > 2$ if we restrict our attention to social choice rules that cannot be manipulated by any individual or by any coalition.

***Theorem 2*:** Given $k \geq 2$, assume that X is a finite set containing at least k+1 alternatives, and the range of $g{:}L(X)^N \to X_k$ is X_k. If no individual or coalition can manipulate g under substitution extension, then some individual is a dictator for g.

Proof: This certainly must be the case if $n = 1$. Theorem 1 establishes the claim for $n = 2$. Suppose than that $n > 2$, $N = \{1,2,...,n\}$, and we have proved the claim for $n-1$.

For each $i \in N$, define the social choice rule g^i for society $N\backslash\{i+1\}$, by setting $g^i(p) = g(r)$ for the profile r obtained by setting $r(h) = p(h)$ for all $h \in N\backslash\{i+1\}$, and setting $r(i+1) = p(i) = r(i)$. (We treat n+1 as person 1.) Because $r(i)$ always equals $r(i+1)$ in the domain of g^i we can think of g^i as defined for a society of $n-1$ persons. Because g cannot be manipulated by any coalition, the contrived rule g^i cannot be manipulated by any coalition. Then the induction hypothesis implies that g^i is dictatorial for each $i \in N$. Suppose that person j is the dictator for g^i and $j \neq i$. We also have $j \neq i + 1$, by definition. We show that person j is a dictator for g.

Choose any profile $p \in L(X)^N$. Let r be any profile such that $r(h) = p(h)$ for $h \in N\backslash\{i,i+1\}$, with $r(i)$ and $r(i+1)$ satisfying

$$r(i) = r(i+1), \text{ and } \; r(i)[m-k+1{:}m] = p(j)[1{:}k] = r(i+1)[m-k+1{:}m]$$

In words, the k alternatives at the top of $p(j)$ are at the bottom of $r(i)$ and $r(i+1)$. We have $g(r) = p(j)[1{:}k]$ because person j is a dictator for g^i. Then $g(r') = p(j)[1{:}k]$ for the profile

$$r' = (r(1),...,r(i-1),r(i),p(i+1),r(i+2),...,r(n))$$

otherwise person i+1 could manipulate g at r. But $g(r') = p(j)[1{:}k]$ implies that $g(p) = p(j)[1{:}k]$ because

$$p = (r(1),...,r(i-1),p(i),p(i+1),r(i+2),...,r(n))$$

which is an i-variant of r′. If g(p) ≠ p(i)[1:k] then person i could manipulate at r′ because r(i) ranks any k-element subset of X to p(j)[1:k], except for p(j)[1:k] itself of course. We have g(p) = p(j)[1:k] for arbitrary p, so person j is a dictator for g.

Suppose, then, that person i is the dictator for g^i for each i ∈ N and m is the cardinality of X. The case m = k is trivial. Hence we may assume that m ≥ k+1. We first take care of the case n = 3. Because m ≥ k+1 we can find two disjoint subsets A and {x,y,z} of X such that A has k−2 members and {x,y,z} has three members. We allow A to be empty (when k = 2.). Let r be the following profile:

Table 11

r(1)	r(2)	r(3)
A	A	A
x	y	z
y	z	x
z	x	y
⋮	⋮	⋮

At r, individuals 1, 2, and 3 may order the members of A differently, but the Table is meant to show that r(1)[1:k−2] = r(2)[1:k−2] = r(3)[1:k−2].

Our first step is to show that A ⊂ g(r). Suppose not. Then y ∈ g(r) because person 1 can get A ∪ {x,z} by reporting r(3) because person 3 is the dictator for g^3. And r(1) ranks A ∪ {x,z} above any k-element subset of X\{y}. We must also have z ∈ g(r) because person 2 can get A ∪ {x,y} by reporting r(1). (Person 1 is the dictator for g^1.) And r(2) ranks A ∪ {x,y} above any k-element subset of X\{z}. Similarly, x must belong to g(r) if A is not a subset of g(r).

Then if A is not a subset of g(r) we have {x,y,z} ⊂ g(r). But g(r) has k members, so there exists some a ∈ A\g(r). But then person 1 can manipulate g at r because person 1 can get A ∪ {x,z} by reporting r(3) (person 3 is the dictator for g^3), and a $\succ_1^r$ y so r(1) ranks A ∪ {x,z} above g(r). Therefore, A ⊂ g(r). If z ∉ g(r) then g(r) = A ∪ {x,y} because person 3 can get A ∪ {x,y} by reporting r(1), because person 3 is the dictator for g^3. And r(3) ranks A ∪ {x,y} above any k-element subset of X\{z}. But if g(r) = A ∪ {x,y} then person 3 can still manipulate at r because person 3 can get A ∪ {y,z} by reporting r(2), because person 2 is the dictator for g^2, and r(3) ranks A ∪ {y,z} above A ∪ {x,y}. Therefore, z belongs to

$g(r)$. But the profile r is symmetrical with respect to the alternatives and the three individuals. So we know that y and z also belong to $g(r)$. Now we have $A \subset g(r)$ and $\{x,y,z\} \subset g(r)$. But this is impossible because $A \cup \{x,y,z\}$ has k+1 elements and $g(r)$ selects k alternatives at each profile in its domain.

We have our proof for $n \leq 3$. Suppose then that $n > 3$. As demonstrated above, if some $j \in N$ is a dictator for some g^i for $i \neq j \neq i+1$ then j is a dictator for g. Suppose that each $i \in N$ is a dictator for g^i. Let r be any profile such that

$$r(1) = r(2),\ r(3) = r(4),\ \text{and } r(1)[1{:}k] \neq r(4)[1{:}k]$$

We have $g(r) = r(1)[1{:}k]$ because person 1 is a dictator for g^1, and $g(r) = r(3)[1{:}k]$ because person 3 is a dictator for g^3. This contradiction concludes the proof. □

The proof of Theorem 2 only uses singleton and two-member coalitions, so it might appear to establish that if no individual or two-person coalition can manipulate g under substitution extension then some individual is a dictator for g. However, the proof employs an induction argument on n, the set of all individuals. We go from n = 2 to n = 3 by merging individuals 2 and 3. Then we go from n = 3 to n = 4 by merging individuals 3 and 4. Therefore, when n = 4 the coalition {3, 4} is in fact the coalition {2, 3, 4}, and so on. Thus, for any n, coalitions with $n-1$ individuals play a role in the proof.

The final theorem assumes that k = 2, but dispenses with the assumption that coalitions of two or more persons cannot manipulate. The assumption that g cannot be manipulated by coalitions containing more than one person is only used in proving Theorem 2. This allowed us to prove that g^i cannot be manipulated. However, if k = 2 we can prove that if g is strategy-proof then so is g^i.

***Theorem* 3**: X is any set with more than two members and n is any positive integer. If $g\colon L(X)^N \to X_2$ is strategy-proof when everyone employs substitution extension, and the range of g is X_2, then g is dictatorial.

Proof: Without loss of generality we consider g^1 as defined in the proof of Theorem 2.

Let p and r be any two profiles in $L(X)^N$ such that

$$p(1) = p(2),\ r(1) = r(2),\ \text{and } p(i) = r(i) \text{ for all } i > 2.$$

Suppose that $g(r)$ ranks above $g(p)$ in the substitution extension of $p(1)$. We will show that g can be manipulated by a single individual.

Given p and r of the previous paragraph, let q be a profile for which q(i) = p(i) for all i > 2, q(1) = q(2), q(1)[1,2] = g(r), q(1)[1:t] = g(r) $\cup$ g(p), where t is the cardinality of the set g(r) $\cup$ g(p). (of course, t = 3 or 4.) We have g(q) = g(r) because g(q(1),r(2),p(3),...,p(n)) = g(r), otherwise person 1 could manipulate at (q(1),p(2),p(3),...,p(n)) via r(1) and thus g(q) = g(r) because person 2 cannot manipulate at r. Therefore, g(q) ranks above g(p) in the substitution extension of p(1).

Suppose that g(p) $\cap$ g(r) = $\emptyset$. Then we have two profiles p and q such that

p(1) = p(2), q(1) = q(2), q(1)[1,2] = g(q), q(1)[1:4] = g(q) $\cup$ g(p)

and p(i) = q(i) for all i > 2.

Let g(p) = {a,b} and g(q) = {x,y}. Without loss of generality, we assume that a $\succ_1^p$ b and x $\succ_1^p$ y. Because the substitution extension of p(1) ranks {x,y} above {a,b} we must have either

$$x \succ_1^p a \succ_1^p y \succ_1^p b \quad \text{or} \quad x \succ_1^p y \succ_1^p a \succ_1^p b.$$

Let s = (q(1),p(1),p(2),...,p(n)). Then s is a 1-variant of p and a 2-variant of q. If g(s) equals {a,b} or {a,y} or {y,b} then person 2 can manipulate at s by reporting q(2). If g(s) = {x,y} or {x,a} or {x,b} then person 1 can manipulate at p by reporting q(1). Therefore, g(s) is not a subset of {a,b,x,y}. On the other hand, if g(s) $\cap$ g(q) = $\emptyset$ then person 1 can manipulate at s by reporting p(1) because the substitution extension of q(1) ranks g(p) above every other two-element subset of X\g(p). Therefore, we must have g(s) = {c,z} for some c $\in$ {x,y,a,b} and some z $\in$ X\{x,y,a,b}. If c $\in$ {a,b} then person 1 can manipulate at s via p(1). Therefore, g(s) = {x,z} or {y,z} for some z $\notin$ g(p) $\cup$ g(q). If g(s) = {x,z} and z $\succ_1^p$ y then person 1 can manipulate at p by reporting q(1) because we would have z $\succ_1^p$ y $\succ_1^p$ b, and we already know that x $\succ_1^p$ a. If g(s) = {x,z} and y $\succ_1^p$ z then person 2 can manipulate at s by reporting q(2), because g(q) = {x,y} and p(2) = p(1). If g(s) = {y,z} and z $\succ_1^p$ x we have z $\succ_1^p$ x $\succ_1^p$ a and thus person 1 can manipulate at p by reporting q(1). (We already have y $\succ_1^p$ b.) If g(s) = {y,z} and x $\succ_1^p$ z then person 2 can manipulate at s by reporting q(2). We have to abandon the supposition that g(p) $\cup$ g(q) has four members, and thus g(p) $\cap$ g(q) contains precisely one alternative, because we know that g(p) and g(q) have at least one member in common.

Because g(p) $\cap$ g(q) is a singleton we can set g(p) = {x,b} and g(q) = {x,y}. We have y $\succ_1^p$ b because the substitution extension of p(1) ranks {x,y} above {x,b}. Set s = (q(1),p(1),p(2),...,p(n)) once again. If g(s) $\cap$ {x,b,y} = $\emptyset$ then person 1 can manipulate at s by reporting p(1). Therefore, we must have $g(s) = \{b,z\}$ or $\{x,z\}$ or $\{y,z\}$ for some $z \in X \backslash \{x,b,y\}$. If $g(s) = \{b,z\}$ then person 2 can manipulate at s by reporting q(2). If $g(s) = \{x,z\}$ and $z \succ_1^p b$ then person 1 can manipulate at p by reporting q(1). If $g(s) = \{x,z\}$ and $b \succ_1^p z$ then $y \succ_1^p b \succ_1^p z$, and hence $y \succ_1^p z$ which implies that person 2 can manipulate at s by reporting q(2). But we also have $y \succ_1^p b$, and hence $g(s) = \{y,z\}$ and $z \succ_1^p x$, which implies that person 1 can manipulate at p by reporting q(1). If $g(s) = \{y,z\}$ then $z \succ_1^p x$, otherwise person 2 could manipulate at s by reporting q(2). Therefore, if $g(s) = \{x,z\}$ we must have $z \succ_1^p x$ and $y \succ_1^p b$, which means that person 1 can manipulate at p by reporting q(1).

We have exhausted all the possibilities. Each leads to manipulation by a single individual. That is, if the substitution extension of p(1) ranks g(r) above g(p) then g can be manipulated by someone at some profile. Since g is strategy-proof, it follows that g^1 is also strategy-proof. Similarly, g^i is strategy proof for all $i \in N$. We know that Theorem 3 is true for n = 2. Suppose $N = \{1, 2, \ldots, n\}$ and Theorem 3 is true for any number of individuals less than n. Because g^i is a strategy-proof rule for a society of $n-1$ individuals the rule g^i is dictatorial for each $i \in N$. Recall that n+1 is interpreted as person 1. Suppose that individual $j \in N \backslash \{i,i+1\}$ is the dictator for g^i. It is easy to show that j is a dictator for g itself. If $p \in L(X)^N$ then $g(p) = p(j)[1,2]$ if the two alternatives in p(j)[1,2] rank at the bottom of p(i) for all $i \in N \backslash \{j\}$ and $p(i) = p(h)$ for all i and h in $N \backslash \{j\}$. Let r be any profile in $\wp$ such that $r(j) = p(j)$, and let $\{p^t\}$ be the standard sequence from p to r. We know that $g(p^0) = g(p) = p(j)[1,2]$. Then for any $t \geq 0$ we have $g(p^{t+1}) = p(j)[1,2]$ if $g(p^t) = p(j)[1,2]$ because at p^t person t+1 ranks any two-element subset of $X \backslash p(j)[1,2]$ above $g(p^t)$. Therefore, $g(p^n) = g(r) = p(j)[1,2]$. Then strategy-proofness of g implies that j is a dictator for g.

Finally, suppose that for each $i \in N$ individual i is the dictator for g^i. If n = 3 then chose three distinct x, y, and z in x and choose $p \in L(X)^N$ such that p has a Latin square involving x, y, and z on top and each individual has the same ordering R of the members of $X \backslash \{x,y,z\}$ as shown in Table 12:

Table 12

p(1)	p(2)	p(3)
x	y	z
y	z	x
z	x	y
R	R	R

At profile p person 2 can get {x, y} by reporting p(1) — because person 1 is the dictator for g^1 — and hence a does not belong to g(p) for any $a \in X\backslash\{x,y,z\}$. But if g(p) = {x, y} then person 3 can manipulate g at p by reporting p(2) — because person 2 is the dictator for g^2. If g(p) = {y, z} then person 1 can manipulate g at p by reporting p(3) — because person 3 is the dictator for g^3. And if g(p) = {x, z} then person 2 can manipulate g at p by reporting p(1) — because person 1 is the dictator for g^1. Therefore, strategy-proofness of g implies that some $j \in N$ is a dictator for some g^i such that $i \neq j \neq i+1$ and hence j is a dictator for g.

Suppose that for each $i \in N$ individual i is the dictator for g^i but $n \geq 4$. Let $p \in L(X)^N$ be the ordering of Table 13:

Table 13

p(1)	p(2)	p(3)	p(4)	···	p(n−1)	p(n)
x	y	z	z	···	z	z
y	z	x	x	···	x	x
z	x	y	y	···	y	y
R	R	R	R	···	R	R

We have g(p) = {x, y} because person 1 is the dictator for g^1, contradicting the conclusion that person 3 is the dictator for g^3, which implies g(p) = {x, z}. Again we conclude that strategy-proofness of g implies that some $j \in N$ is a dictator for some g^i such that $i \neq j \neq i+1$ and hence j is a dictator for g. □

Theorem 2, which establishes dictatorship for any number n of individuals and any number k of chosen alternatives general n and general k required non-manipulability by *coalitions* as well as individuals. The requirement of invulnerability of manipulation by coalitions can be dropped if either n = 2(Theorem 1) or k = 2 (Theorem 3). We do not know of if the statement of Theorem 2 remains true if we drop the assumption of invulnerability of manipulation by coalitions.

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